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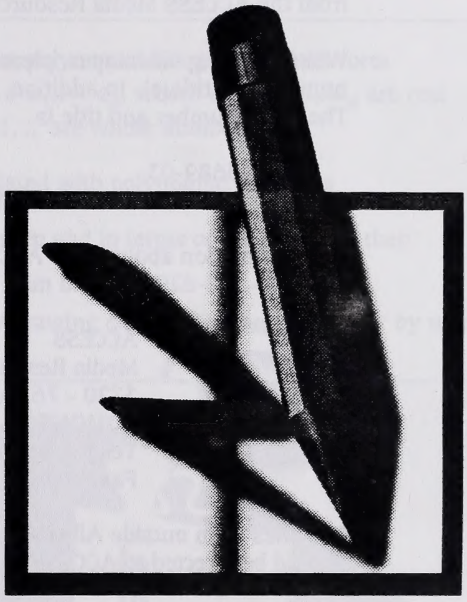
POLYNOMIAL FUNCTIONS



Lesson 1: Introduction to Polynomial Functions and Their Graphs

Student Workbook

Produced by

THE EDUCATION STATION

for

Alberta EDUCATION

This guide accompanies **Introduction to Polynomial Functions and Their Graphs** from the Math Factor series.

Writer: Gordon Mallett
Editor: Wilma Gonsalves
Producer: Katherine Rankin

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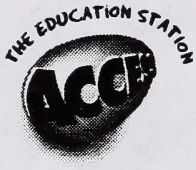

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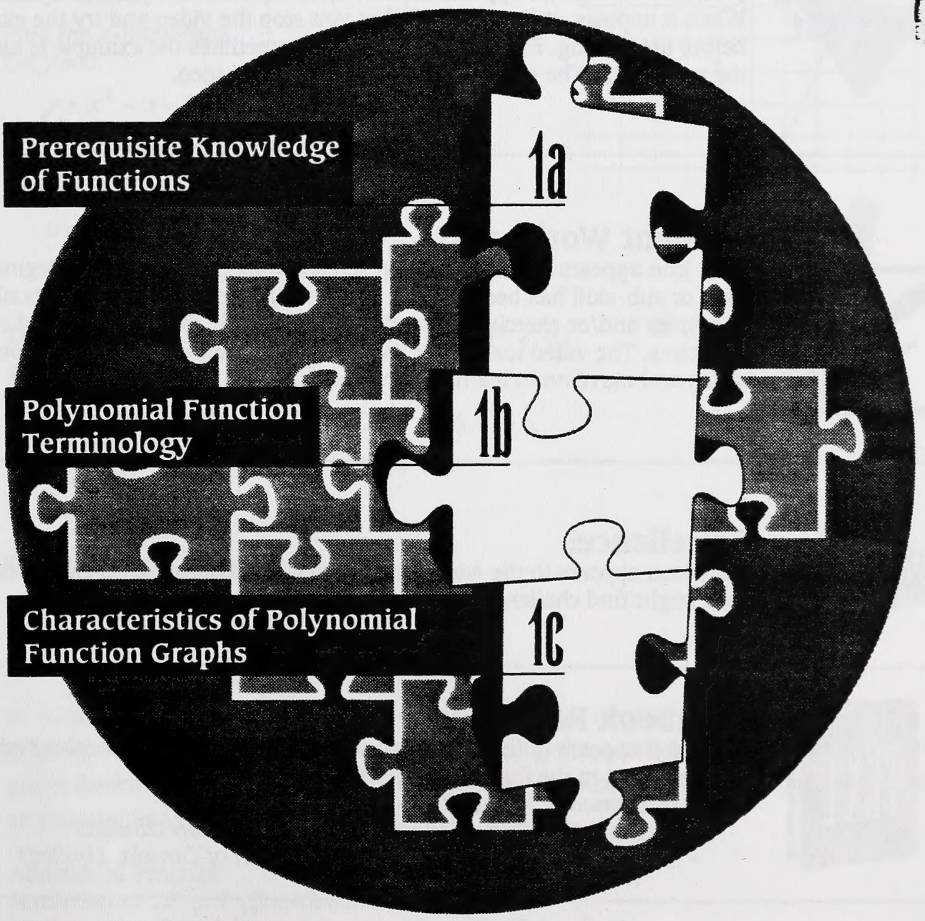
Lesson 1: INTRODUCTION TO POLYNOMIAL FUNCTIONS AND THEIR GRAPHS

LEARNER EXPECTATIONS

By the end of Lesson 1 you should be able to

- demonstrate an understanding that a polynomial function is of the form $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0$, where $a_0, a_1, ..., a_n$ are real numbers and the exponents $n, n-1, ...$ are whole numbers
- correctly use the terminology associated with polynomial functions
- classify polynomial functions by degree and in terms of the nature of their coefficients
- investigate polynomial functions by drawing sketches of their graphs or by using calculators or computers

This lesson has three parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

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- *Mathematics 12 (Addison-Wesley)*
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- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to draw and describe the graph of a quadratic function? Before watching this program, refresh your memory by studying the following example.

Example

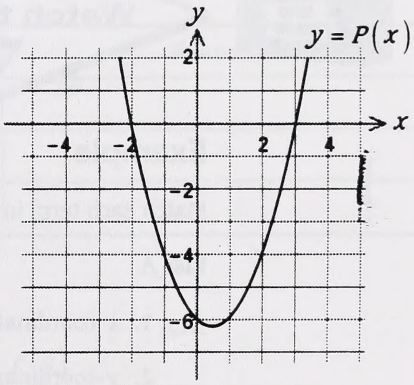
Sketch the graph of the quadratic function $y = P(x)$, where $P(x) = x^2 - x - 6$ by finding the y-intercept, the x-intercept and the vertex. Summarize these points in a table of values; then describe the graph.

Solution

Find the y-intercept:
Let $x = 0$.

$$y = x^2 - x - 6$$
$$y = (0)^2 - (0) - 6$$
$$= -6$$

The y-intercept is at the point $(0, -6)$.



Find the x-intercepts:
Let $y = 0$.

$$y = x^2 - x - 6$$
$$0 = x^2 - x - 6$$
$$0 = x^2 - x - 6$$
$$0 = (x - 3)(x + 2)$$
$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = 3 \qquad\qquad x = -2$$

The x-intercepts are at the points $(3, 0)$ and $(-2, 0)$.

Find the vertex.

$$x = \frac{1}{2}[3 + (-2)] \qquad y = (0.5)^2 - (0.5) - 6$$
$$= \frac{1}{2}(1) \qquad\qquad = -6.25$$
$$= 0.5$$

The vertex is $(0.5, -6.25)$.

x	0	3	-2	0.5
y	-6	0	0	-6.25

In quadrant 2, the graph of $y = x^2 - x - 6$ decreases, crossing the x-axis at -2. The graph continues to decrease in quadrant 3, crossing the y-axis at -6. In quadrant 4, the graph decreases until it reaches $(0.5, -6.25)$; then it increases, crossing the x-axis at 3 and continues to increase in quadrant 1.

TEXT

Additional Practice
Mathematics 12: p. 7 (Questions 6a, 6d)
Mathematics: Principles and Process 12: p. 48 (Example 1)
Holtmath 12: p. 27 (Question 8)

Prerequisite Knowledge of Functions

Definition
A function is a set of ordered pairs in which every element of the domain maps into only one element of the range.

It is important that you understand the meaning of the terms listed at the right when working with problems involving functions. Watch for them in Video Segment 1a.

- x-coordinate
- y-coordinate
- domain
- range
- function notation
- vertical line test
- trinomial
- polynomial
- degree
- zeros

MATH FACTOR

Watch the Introduction and Video Segment 1a.

Example

Match each term in List A with the phrase in List B that best describes it.

List A

___ 1. x-coordinate
___ 2. y-coordinate
___ 3. domain
___ 4. range
___ 5. vertical line test
___ 6. trinomial
___ 7. polynomial
___ 8. degree
___ 9. zeros

List B

a. the set of all the first members of ordered pairs represented by a graph
b. where no two points on a graph can be joined by a vertical line, so the graph represents a function
c. the highest sum of exponents on the variables of any term in a polynomial
d. an expression containing three terms
e. the first coordinate in an ordered pair
f. values of x in a function for which $f(x) = 0$
g. an expression containing one or more terms
h. the set of all the second members of ordered pairs represented by a graph
i. the second coordinate in an ordered pair

Solution

1e, 2i, 3a, 4h, 5b, 6d, 7g, 8c, 9f

TEXT

Additional Examples
Mathematics 12: pp. 1-6 (Examples 1-3), p. 9 (Examples 1-3)
Holtmath 12: pp. 22, 23

Exercises

1. Classify the following functions according to degree and number of terms.

Function	Degree	Number of Terms
$P(x) = 3x^2 - x + 10$		
$f(x) = -2x - 6$		
$h(x) = 3x^3 - 5$		
$g(x) = 10$		

2. Indicate whether the following equations are equations of fourth-degree functions by choosing **yes** or **no**. Give a reason.

Expression	Degree 4 Function (yes or no)	Reason
$y^2 = x^4 - 5$		
$y = x^4 - x^5$		
$y = (x^2 + 1)(1 - x^2)$		

3. If $f(x) = 3x^2 + x - 2$, find the following:

   a. $f(-2)$ b. $f(a)$

EXERCISE 1a SOLUTIONS are at the back of this workbook.

TEXT

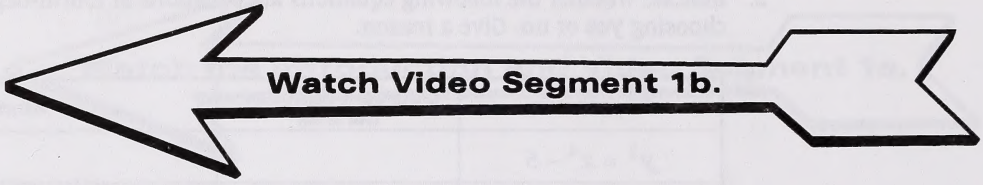
Additional Practice
Mathematics 12: p. 6 (Questions 2-4), p. 11 (Questions 5, 10)
Holtmath 12: p. 91 (Question 7)

Polynomial Function Terminology

Definition

A polynomial function is a set of ordered pairs of the form $(x, f(x))$ which is of the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers and the exponents $n, n-1, \ldots$ are whole numbers.

Several mathematical terms are required to explain the classification of polynomial functions. Watch and listen carefully for these terms in Video Segment 1b.



The Polynomial Function $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$

- a_n is the leading or initial coefficient.
- a_0 is the constant term. It also represents the y-intercept when you look at the graph of $y = f(x)$.
- n is the degree of the polynomial.

Number Systems

- natural numbers, N
  $\{1, 2, 3, \ldots\}$
- whole numbers, W
  These include natural numbers.
  $\{0, 1, 2, \ldots\}$
- integers, I
  These include whole numbers.
  $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- rational numbers, Q
  These include integers.
  $\left\{\frac{a}{b} \middle| a, b \in I, b \neq 0\right\}$
- irrational numbers, $\overline{Q}$
  These are numbers that cannot be written in the form $\left\{\frac{a}{b} \middle| a, b \in I, b \neq 0\right\}$.
  For example, $\sqrt{2}$, $\sqrt{5}$, and π.
- real numbers, R
  These include rational and irrational numbers.
- imaginary numbers
  These numbers contain the square roots of negative numbers, such as $\sqrt{-16}$ (which can be written as $\sqrt{16i^2} = \pm 4i$, where $i^2 = -1$).

Classification of Polynomial Functions Based on the Nature of the Coefficients

Type of Polynomial Function	Description	Example
integral polynomial	All the coefficients are integers.	$f(x) = 2x^2 + 3x + 6$
rational polynomial	All the coefficients are rational.	$f(x) = 2x^2 + \frac{3}{4}x + \frac{1}{2}$
real polynomial	All the coefficients are real.	$f(x) = \sqrt{2}x^2 + 3x + \frac{1}{2}$

Note: Integral and rational polynomial functions can also be classified as real polynomial functions.

Exercises

4. Indicate whether the expressions are polynomial functions by choosing **yes** or **no**. If **no**, give a reason.

Expression	Polynomial Function (yes or no)	Reason
$g(x) = \sqrt{5}x^{-1} + 10$	no	-1 is not a whole number exponent.
$f(x) = x^5 - \sqrt{2}$		
$h(x) = x^5 + \sqrt{-2}$		
$f(x) = -x^4 + x^{0.5}$		
$j(x) = (x+1)(x-2)$		

5. Classify the polynomial functions according to their coefficients and give a reason for your answer.

Function	Coefficient Classification	Reason
$f(x) = 8$	integral polynomial function	8 is an integer.
$g(x) = 8x^2 + \frac{1}{2}$		
$f(x) = \sqrt{5}x$		
$f(x) = \sqrt{10}$		
$h(x) = 3x^2 + \sqrt{9}x$		
$f(x) = x^3 + \sqrt{2}x^2 + \frac{3}{4}$		

EXERCISE 1b SOLUTIONS are at the back of this workbook.

TEXT

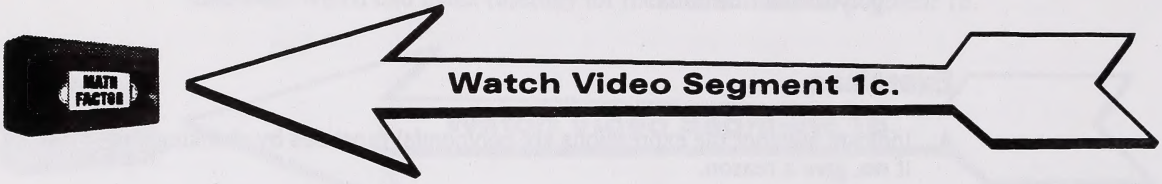
Additional Practice
Holtmath 12: p. 91 (Questions 2, 3)

Characteristics of Polynomial Function Graphs

When drawing sketches of the graphs of polynomial functions, it is important to keep in mind:

- the x- and y- intercepts
- the degree of the polynomial function
- the sign of the leading coefficient (i.e., the coefficient of the term of highest degree)

These factors are discussed in Video Segment 1c.



The x- and y- intercepts

- There is always one y-intercept (which is the constant term of the polynomial).
- The x-intercepts are the zeros of the function.
- Odd-degree polynomials must have at least one x-intercept.

Degree of the Polynomial Function

- The number of intervals of increase or decrease is **at most** equal to the degree of the function.

Sign of the Leading Coefficient

- If the sign is **positive**, then to the far right, in quadrant 1, the function increases.
- If the sign is **negative**, then to the far right, in quadrant 4, the function decreases.

Example 1

Find and sketch the x- and y-intercepts for the graph of the function $y = f(x)$, where $f(x) = x^3 - x^2 - 4x + 4$.

Solution

To find the y-intercept, let $x = 0$.

$$\begin{aligned} f(x) &= 0^3 - 0^2 - 4(0) + 4 \\ &= 4 \end{aligned}$$

The y-intercept is at the point $(0, 4)$.

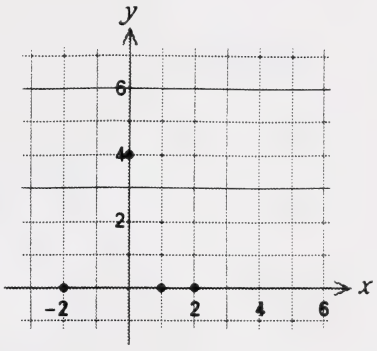
To find the x-intercepts, factor the polynomial; then let $f(x) = 0$.

$$\begin{aligned} f(x) &= x^3 - x^2 - 4x + 4 \\ &= x^2(x-1) - 4(x-1) \\ &= (x-1)(x^2-4) \\ &= (x-1)(x+2)(x-2) \end{aligned}$$

$$0 = (x-1)(x+2)(x-2)$$

$x - 1 = 0$ $\quad$ $x + 2 = 0$ $\quad$ $x - 2 = 0$

$x = 1$ $\quad$ $x = -2$ $\quad$ $x = 2$



The x-intercepts are at the points $(1,0)$, $(-2,0)$ and $(2,0)$.

Example 2

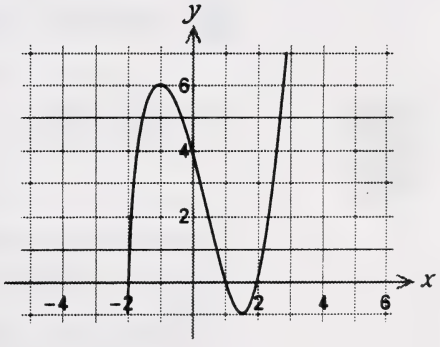
Use the equation of the polynomial function in Example 1 to determine the number of intervals of increase or decrease and the sign of the leading coefficient and then complete the sketch.

Solution

The graph must pass through the four points identified in Example 1.

The leading coefficient is positive; thus, the graph rises to the right in quadrant 1.

The degree is 3, and in this case, hence there is a maximum of 3 intervals of increase or decrease.



TEXT

Additional Examples
Mathematics 12: pp. 42-43, p. 72

Exercises

6. Sketch the graph of $y = P(x)$, where $P(x) = (x-1)(x+1)(x-2)(x+2)$.

7. Use a table of values to help sketch the graph of $y = P(x)$, where $P(x) = -(x+2)^3$ for the interval $-4 \le x \le 0$.

EXERCISE 1c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 45 (Questions 6-8), p. 77 (Questions 2-4)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. You should be familiar with polynomial function terminology and graphing skills before continuing on to **Lesson 2: Division of Polynomial Functions**.

Solutions

EXERCISE 1a

1.

Function	Degree	Number of Terms
$P(x)=3x^2-x+10$	2	3 (trinomial)
$f(x)=-2x-6$	1	2 (binomial)
$h(x)=3x^3-5$	3	2 (binomial)
$g(x)=10$	0	1 (monomial)

Note that $10=10x^0$.

2.

Expression	Degree 4 Function (yes or no)	Reason
$y^2=x^4-5$	no	not a function because of y^2
$y=x^4-x^5$	no	degree 5 function
$y=(x^2+1)(1-x^2)$	yes	$y=x^2-x^4+1-x^2$

3. a. $f(x)=3x^2+x-2$

$f(-2)=3(-2)^2+(-2)-2$

$=3(4)-2-2$

$=12-4$

$=8$

b. $f(x)=3x^2+x-2$

$f(a)=3(a)^2+(a)-2$

$=3a^2+a-2$

EXERCISE 1b

4.

Expression	Polynomial Function (yes or no)	Reason
$g(x)=\sqrt{5}x^{-1}+10$	no	-1 is not a whole number exponent.
$f(x)=x^5-\sqrt{2}$	yes	
$h(x)=x^5+\sqrt{-2}$	no	$\sqrt{-2}$ is not a real number.
$f(x)=-x^4+x^{0.5}$	no	0.5 is not a whole number.
$j(x)=(x+1)(x-2)$	yes	

5.

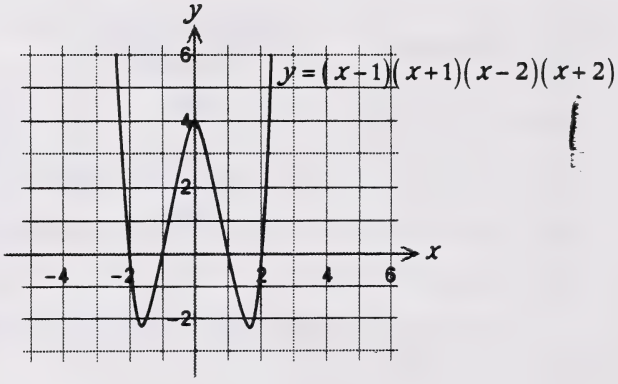
Function	Coefficient Classification	Reason
$f(x) = 8$	integral polynomial function	8 is an integer.
$g(x) = 8x^2 + \frac{1}{2}$	**rational polynomial function**	$\frac{1}{2}$ **is a rational number.**
$f(x) = \sqrt{5}x$	**real polynomial function**	$\sqrt{5}$ **is an irrational number.**
$f(x) = \sqrt{10}$	**real polynomial function**	$\sqrt{10}$ **is an irrational number.**
$h(x) = 3x^2 + \sqrt{9}x$	**integral polynomial function**	$\sqrt{9}$ **is the integer** ± 3.
$f(x) = x^3 + \sqrt{2}x^2 + \frac{3}{4}$	**real polynomial function**	**All coefficients are real.**

EXERCISE 1c

6. $P(x) = (x-1)(x+1)(x-2)(x+2)$

y-intercept = 4
x-intercepts = 1, −1, 2, −2

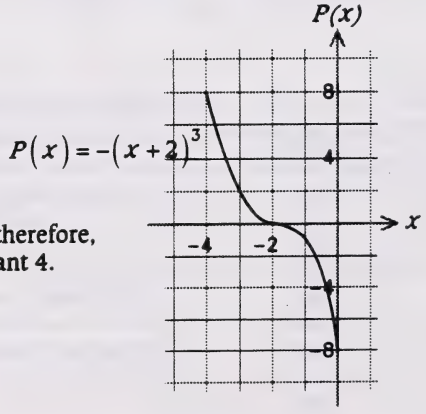
The leading coefficient is positive; therefore, the graph rises to the right in quadrant 1.



7. $P(x) = -(x+2)^3$, $-4 \le x \le 0$

x	−4	−3	−2	−1	0
y	8	1	0	−1	−8

The leading coefficient is negative; therefore, the graph falls to the right in quadrant 4.



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POLYNOMIAL FUNCTIONS

Lesson 2:
Division of Polynomial Functions

Student Workbook

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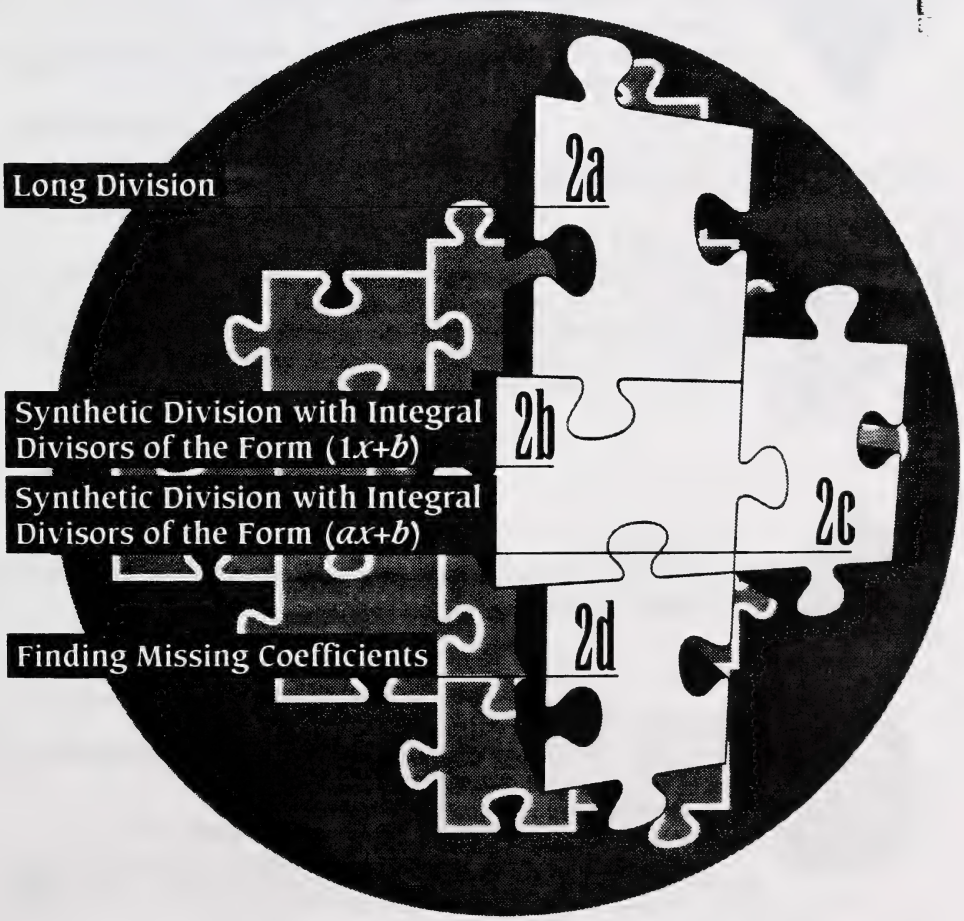
Lesson 2: DIVISION OF POLYNOMIAL FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 2 you should be able to

- divide integral polynomials by a binomial divisor using long division
- divide integral polynomials by a binomial linear divisor using synthetic division
- use the division algorithm to write the division operation in the form $P(x) = Q(x) \bullet D(x) + R(x)$
- determine missing coefficients in the divisor, dividend or quotient

This lesson has four parts:



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Building Blocks

Do you remember how to subtract and multiply polynomials? Before watching this program, refresh your memory by studying the following examples and doing the exercises.

Example 1

Subtraction of Polynomials

One of the steps of long division with polynomials involves the subtraction of polynomials. Recall that subtraction is the same as adding the additive inverse; for example, $5-17=5+(-17)$.

To subtract polynomials, change the second polynomial to its additive inverse; then add the two polynomials together.

Subtract	Add the additive inverse
$3x^2-4x-7$	$3x^2-4x-7$
$\underline{5x^2-3x+2}$	$\underline{-5x^2+3x-2}$
	$-2x^2-1x-9$

Example 2

Multiplication of Polynomials

Checking the division of polynomials involves the multiplication of polynomials, such as multiplying a binomial by a trinomial.

$$(2x-3)(4x^2-x+1)=2x(4x^2-x+1)-3(4x^2-x+1)$$

Multiply the second polynomial first by $2x$ and then by -3.

$$=8x^2-2x^2+2x-12x^2+3x-3$$

$$=8x^3-14x^2+5x-3$$

Collect like terms.

Exercises

1. Subtract the following polynomials.

   a. $2x^2-4x+5$ minus $3x^2+6x-4$

   b. $9x^2-4x+3$ minus $-7x^2+9x-4$

2. Expand and simplify the following:

   a. $(x+3)(2x^2-7x+4)+10$

   b. $3(2x-1)^2-4(x+6)(x-7)$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Long Division

Example

Divide $\left(3x^3+4x-5\right)$ by $\left(x-1\right)$.

Solution

$$
\begin{array}{r}
3x^2+3x+7 \\
x-1\overline{\left)\begin{array}{r} 3x^3+0x^2+4x-5 \\ \underline{3x^3-3x^2} \\ 3x^2+4x \\ \underline{3x^2-3x} \\ 7x-5 \\ \underline{7x-7} \\ 2 \end{array}\right.}
\end{array}
$$

Check

$$
\begin{aligned}
P(x) &= D(x)\bullet Q(x)+R(x) \\
&= (x-1)\left(3x^2+3x+7\right)+2 \\
&= 3x^3+3x^2+7x-3x^2-3x-7+2 \\
&= 3x^3+4x-5
\end{aligned}
$$

Steps for Doing Long Division

Step 1: **Arrange the terms** of the divisor and dividend in descending order of degree. Use zeros to replace any missing terms.

Step 2: **Divide** the first term of the dividend by the first term of the divisor. The result, $3x^2$, becomes the first term of the quotient.

Step 3: **Multiply** the term in the quotient times each term of the divisor. Write these terms under the corresponding terms of the dividend.

Step 4: In order to **subtract**, the first terms must be the same.

Step 5: **Bring down** the next term. (In this case it would be $+4x$.)

Step 6: Continue **dividing**, **multiplying**, and **subtracting** (Steps 2 to 4) until the degree of the remainder is less than the degree of the divisor.

Challenge yourself

Divide $\left(3x^2+2x^3-4\right)$ by $\left(2x-1\right)$.

MATH FACTOR

Watch the Introduction and Video Segment 2a.

TEXT

Additional Examples
Mathematics 12: pp. 49, 50
Holtmath 12: p. 92

Exercises

1. The division algorithm can be used to check your division. If the dividend is represented by $P(x)$, the divisor by $Q(x)$, the quotient by $Q(x)$ and the remainder by $R(x)$, then rewrite the division algorithm in functional notation.

2. If $P(x) = D(x) \bullet Q(x) + R(x)$, then find an expression for $P(x)$ when $R(x) = -3$, $Q(x) = x^2 + 2x + 1$ and $D(x) = 2x - 1$.

3. Since division by zero is undefined, the divisor can never equal zero. State the **nonpermissible values** for x (sometimes called the **restrictions**) in the following divisions.

   a. $\left(3x^2 + 3x + 7\right) \div (x - 2)$

   b. $\left(8x^3 - 27\right) \div (2x + 3)$

4. Divide and identify the restrictions on the variable. Check your solutions.

   a. $\left(3x^3 - 2x^2 - 7\right) \div \left(x - 3\right)$

   b. $\left(-5x - 8 + 6x^2\right) \div \left(2x - 1\right)$

5. For $\left(12x^4 - 39x^2 + 8x - 8x^3 + 12\right) \div \left(2x^2 - 3x - 2\right)$, find $Q(x)$ and $R(x)$.

EXERCISE 2a SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 52 (Questions 3, 7, 9)
Holtmath 12: p. 93 (Questions 6, 10, 11)

Synthetic Division with Integral Divisors of the Form $(x \pm 1)$

In order to use synthetic division, the divisor must be linear, which means that it can be written in the form $ax + b$. In this segment, the leading coefficient of the divisor will be restricted so that $a = 1$.

Complete the following long divisions on the left side of this page.

Long Division ⟷ **Synthetic Division**

$x+2\overline{)\,4x^4+0x^3-11x^2+8x-6}$

$+2 \mid 4 \quad 0 \quad -11 \quad 8 \quad -6$

Challenge yourself

$x-3\overline{)\,2x^5-5x^4-3x^3-6x^2+20x-16}$

Watch Video Segment 2b.

Now redo the previous equations using synthetic division.

Example 1

Divide $\left(2x^4 - 7x^3 + 5x + 4\right)$ by $\left(x - 2\right)$ using synthetic division.

Long Division ⟷ **Synthetic Division**

$$\left(x-2\right)\overline{)\,2x^4 - 7x^3 + 0x^2 + 5x + 4}$$

$$\begin{array}{r|ccccc} -2 & 2 & -7 & 0 & 5 & 4 \\ & & -4 & 6 & 12 & 14 \\ \hline & 2 & -3 & -6 & -7 & -10 \end{array}$$

(row A)
(row B)
(row C)

Since the divisor is linear, the quotient is one degree less than the dividend.
$Q(x) = 2x^3 - 3x^2 - 6x - 7,\ R(x) = -10$

Steps for Doing Synthetic Division

Step 1: Rewrite the dividend and divisor in **descending order of degrees** inserting zeros for any missing terms.

Step 2: Write the **constant** term of the divisor and all of the **coefficients** of the dividend in row A as shown.

Step 3: **Bring down** the leading coefficient into row C.

Step 4: **Multiply** the constant in the divisor times the number in row C. Write this product in row B under the next term in row A.

Step 5: **Subtract** the numbers in rows A and B and write their difference in row C.

Step 6: **Continue multiplying and subtracting** (Step 4 and Step 5) until each column is completed. The last number in row C is the remainder.

TEXT

Additional Examples
Mathematics 12: p. 51
Mathematics: Principles and Process 12: p. 149
Holtmath 12: p. 96

Exercises

6. Using synthetic division, determine the quotient and the remainder for each of the following:

   a. $\left(x^5 - 2x^3 - 4x^2 - 6x + 12\right) \div \left(x - 3\right)$

   b. $\left(-5x^3 + 2x^2 - 3\right) \div \left(x + 2\right)$

EXERCISE 2b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 52 (Question 6)
Mathematics: Principles and Process 12: p. 150 (Questions 2, 3)
Holtmath 12: p. 93 (Questions 6a, 6d, 6e, 6f, 6g)

EXCELLENCE

Example 2

Divide $\left(2x^4 - 7x^3 + 5x + 4\right)$ by $\left(x - 2\right)$ using the addition method of synthetic division.

Solution

Using the **addition method**, the constant of the divisor is changed to its additive inverse (by multiplying it by -1). Multiplying by the additive inverse of the divisor changes all the signs in row B. Thus, you can add rows A and B.

Subtraction Method ◄——— **Addition Method**
[row A – row B = row C] [row A + row B = row C]

−2	2	−7	0	5	4	(row A)	+2	2	−7	0	5	4	(row A)
		−4	6	12	14	(row B)			+4	−6	−12	−14	(row B)
	2	−3	−6	−7	−10	(row C)		2	−3	−6	−7	−10	(row C)

$Q(x) = 2x^3 - 3x^2 - 6x - 7, \; R(x) = -10$

Synthetic Division with Integral Divisors of the Form $(ax - b)$

Use synthetic division to divide $\left(6x^3 + x^2 - 1\right)$ by $\left(2x - 4\right)$.

Watch Video Segment 2c.

Example

Divide $\left(2x^2 + 3x - 6\right)$ by $\left(2x - 1\right)$ using long division. Divide the divisor by 2; then use this smaller divisor to divide into the original dividend. How are $Q(x)$ and $R(x)$ affected when $D(x)$ is divided by 2?

Solution

Divisor's leading coefficient ≠ 1

$$\begin{array}{r} x+2 \\ 2x-1 \overline{)\ 2x^2+3x-6} \\ \underline{2x^2-1x} \\ 4x-6 \\ \underline{4x-2} \\ -4 \end{array}$$

This cannot be done by synthetic division.

Divisor's leading coefficient = 1

$$\begin{array}{r} 2x+4 \\ x-\frac{1}{2} \overline{)\ 2x^2+3x-6} \\ \underline{2x^2-1x} \\ 4x-6 \\ \underline{4x-2} \\ -4 \end{array}$$

If $D(x)$ is divided by 2, then $Q(x)$ is twice as big, but $R(x)$ is unchanged.

This can be done by synthetic division.

$$\begin{array}{r|rrr} -\frac{1}{2} & 2 & 3 & -6 \\ & & -1 & -2 \\ \hline & 2 & 4 & -4 \end{array}$$

Using the **division algorithm**, you can factor out the 2 from the quotient and multiply it back into the divisor to express the division in its original form.

$$P(x) = D(x) \bullet Q(x) + R(x)$$

$$= \left(x - \frac{1}{2}\right) \bullet (2x + 4) - 4 \longrightarrow D(x) = \left(x - \frac{1}{2}\right);\ Q(x) = (2x + 4);\ R(x) = -4$$

$$= \left(x - \frac{1}{2}\right) \bullet 2(x + 2) - 4$$

$$= 2\left(x - \frac{1}{2}\right) \bullet (x + 2) - 4$$

$$= (2x - 1) \bullet (x + 2) - 4 \longrightarrow D(x) = (2x - 1);\ Q(x) = (x + 2);\ R(x) = -4$$

Exercises

7. Divide $\left(4x^4 + 7x^3 - 14x^2 - 13x + 4\right)$ by $(4x - 1)$ using synthetic division. State the quotient and the remainder.

EXERCISE 2c SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 53 (Question 12)
Mathematics: Principles and Process 12: p. 150 (Questions 4c, 4d, 5c, 5e)
Holtmath 12: p. 93 (Questions 6c, 11c)

Finding Missing Coefficients

Watch Video Segment 2d.

Use your skills in long division and synthetic division to find missing coefficients in a polynomial expression.

Exercise

8. When $\left(6x^3 + kx^2 - 1\right)$ is divided by $\left(2x - 1\right)$, the quotient is $\left(3x^2 + mx + 1\right)$ and the remainder is 0. Determine the values of the coefficients k and m.

EXERCISE 2d SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 53 (Questions 17, 18)
Mathematics: Principles and Process 12: p. 150 (Question 7)
Holtmath 12: p. 93 (Question 13)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of this lesson and check off the points that you have mastered. Division becomes a building block or tool for learning the concepts in the remaining lessons of this module. You should be skillful with division before you continue on to **Lesson 3: The Remainder and Factor Theorems**.

Solutions

BUILDING BLOCKS

1. a. Subtract ⟶ Add

$$\begin{array}{rr} 2x^2-4x+5 & 2x^2-4x+5 \\ \underline{3x^2+6x-4} & \underline{-3x^2-6x+4} \\ & -x^2-10x+9 \end{array}$$

b. Subtract ⟶ Add

$$\begin{array}{rr} 9x^2-4x+3 & 9x^2-4x+3 \\ \underline{-7x^2+9x-4} & \underline{+7x^2-9x+4} \\ & 16x^2-13x+7 \end{array}$$

2. a. $(x+3)(2x^2-7x+4)+10 = 2x^3-7x^2+4x+6x^2-21x+12+10$

$$= 2x^3-x^2-17x+22$$

b. $3(2x-1)^2-4(x+6)(x-7) = 3(2x-1)(2x-1)-4(x+6)(x-7)$

$$= 3(4x^2-2x-2x+1)-4(x^2-7x+6x-42)$$
$$= 3(4x^2-4x+1)-4(x^2-x-42)$$
$$= 12x^2-12x+3-4x^2+4x+168$$
$$= 8x^2-8x+171$$

EXERCISE 2a

1. Division Algorithm: Dividend = [Divisor] • [Quotient] + Remainder

$$P(x) = D(x) \bullet Q(x) + R(x)$$

2. $P(x) = (2x-1)(x^2+2x+1)-3$

$$= 2x^3+4x^2+2x-x^2-2x-1-3$$
$$= 2x^3+3x^2-4$$

3. a. $(x-2) \neq 0$; therefore, $x \neq 2$.

b. $(2x+3) \neq 0$; therefore, $x \neq -\frac{3}{2}$.

4. a.

$$\begin{array}{r} 3x^2+7x+21,\ x \neq 3 \\ x-3\overline{)\ 3x^3-2x^2+0x-7} \\ \underline{3x^3-9x^2} \\ 7x^2+0x \\ \underline{7x^2-21x} \\ 21x-7 \\ \underline{21x-63} \\ 56 \end{array}$$

Check

$$P(x) = (x-3)(3x^2+7x+21)+56$$
$$= 3x^3+7x^2+21x-9x^2-21x-63+56$$
$$= 3x^3-2x^2-7$$

b. $2x-1 \overline{)\, 6x^2-5x-8}$ $\quad 3x-1,\ x \neq \frac{1}{2}$

$$\begin{array}{r} 6x^2-3x \\ \hline -2x-8 \\ -2x+1 \\ \hline -9 \end{array}$$

Check

$$\begin{aligned} P(x) &= (2x-1)(3x-1)-9 \\ &= 6x^2-2x-3x+1-9 \\ &= 6x^2-5x-8 \end{aligned}$$

5. $2x^2-3x-2 \overline{)\, 12x^4-8x^3-39x^2+8x+12}$ $\quad 6x^2+5x-6$

$$\begin{array}{r} 12x^4-18x^3-12x^2 \\ \hline 10x^3-27x^2+8x \\ 10x^3-15x^2-10x \\ \hline -12x^2+18x+12 \\ -12x^2+18x+12 \\ \hline 0 \end{array}$$

$Q(x) = 6x^2+5x-6$

$R(x) = 0$

$D(x) = 2x^2-3x-2,\ D(x) \neq 0$

$= (2x+1)(x-2)$

Therefore, $x \neq 2, -\frac{1}{2}$.

EXERCISE 2b

6. a.

-3	1	0	-2	-4	-6	12
		-3	-9	-21	-51	-135
	1	3	7	17	45	147

$Q(x) = x^4+3x^3+7x^2+17x+45,\ R(x) = 147$

b.

$+2$	-5	2	0	-3
		-10	24	-48
	-5	12	-24	45

$Q(x) = -5x^2+12x-24,\ R(x) = 45$

EXERCISE 2c

7. $4x-1 \overline{)\, 4x^4+7x^3-14x^2-13x+4}$

$$\begin{array}{r} x^3+2x^2-3x-4 \\ \hline 4x^4-1x^3 \\ \hline 8x^3-14x^2 \\ 8x^3-2x^2 \\ \hline -12x^2-13x \\ -12x^2+3x \\ \hline -16x+4 \\ -16x+4 \\ \hline 0 \end{array}$$

$$\begin{array}{r|rrrrr} -\frac{1}{4} & 4 & 7 & -14 & -13 & 4 \\ & & -1 & -2 & 3 & 4 \\ \hline & 4 & 8 & -12 & -16 & 0 \end{array}$$

$$P(x)=\left(x-\frac{1}{4}\right)\left(4x^3+8x^2-12x-16\right)+0$$
$$=4\left(x-\frac{1}{4}\right)\left(x^3+2x^2-3x-4\right)$$
$$=(4x-1)\left(x^3+2x^2-3x-4\right)$$

$$Q(x)=x^3+2x^2-3x-4;\ R(x)=0$$

EXERCISE 2d

8. $2x-1 \overline{)\, 6x^3+kx^2+0x-1}$

$$\begin{array}{r} 3x^2+mx+1 \\ \hline 6x^3-3x^2 \\ \hline (k+3)x^2+0x \\ 2mx^2-mx \\ \hline mx-1 \\ 2x-1 \\ \hline 0 \end{array}$$

$(k+3)-2m=0$

$k=2m-3$

$m-2=0$

$\therefore m=2$

If $m=2$, $k=2(2)-3$, then $k=1$.

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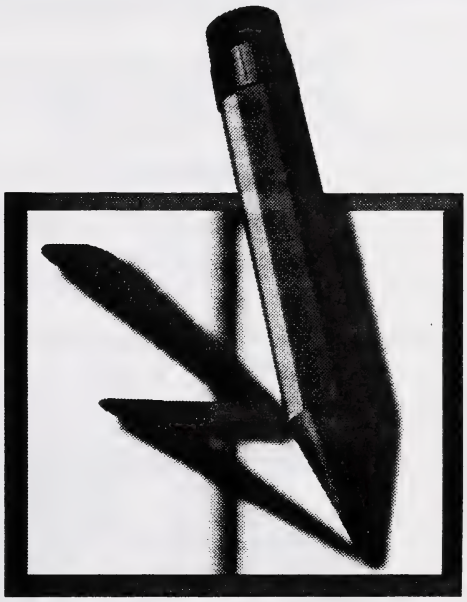
MATH FACTOR

POLYNOMIAL FUNCTIONS



Lesson 3:
The Remainder and Factor Theorems

Student Workbook

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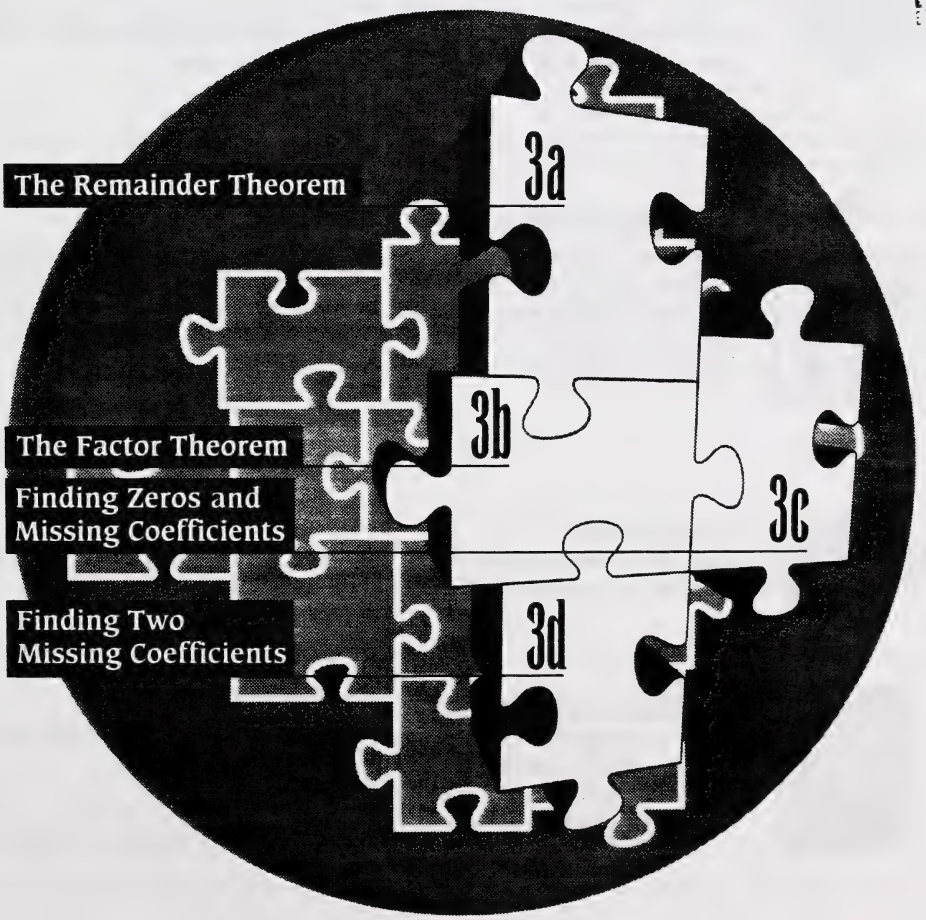
Lesson 3: THE REMAINDER AND FACTOR THEOREMS

LEARNER EXPECTATIONS

By the end of Lesson 3 you should be able to

- use the Remainder Theorem to evaluate polynomial functions
- use the Factor Theorem to determine factors of a polynomial
- prove both the Remainder and Factor Theorems
- apply these theorems to problems that involve finding zeros, finding missing coefficients and sketching graphs

This lesson has four parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember how to factor simple polynomials, solve systems of equations involving two unknowns, and evaluate a function using function notation? Before watching this program, refresh your memory by studying the following examples and doing the exercises.

Factoring Polynomials

There are a few forms of polynomials that you should recognize when factoring:

- Common Factor

$$6x^4 - 3x^2 + 12x = 3x\left(2x^3 - x + 4\right)$$

Always look first for the greatest common factor.

- Difference of Squares

$$4y^2 - 25 = (2y - 5)(2y + 5)$$

Look for a binomial with two perfect squares subtracted.

- Second-Degree Trinomials

$$a^2 + 7a - 30 = (a - 3)(a + 10)$$

The sum of -3 and 10 is 7, and their product is -30.

$$2x^2 + 5x - 12 = (2x - 3)(x + 4)$$

Be careful when x^2 has a coefficient not equal to one.

- Perfect Squares

$$r^2 - 10r + 25 = (r - 5)(r - 5)$$
$$= (r - 5)^2$$

A special case of factoring trinomials occurs when the constant term is the square of half the first-degree term.

Solving Systems of Equations with Two Unknowns

$$3a - 2b + 18 = 0$$
$$2a + 5b = 7$$

$3a - 2b = -18$ (Multiply by 2.)

$2a + 5b = 7$ (Multiply by 3.)

$$6a - 4b = -36$$
$$\underline{6a + 15b = 21} \quad \text{(subtract)}$$
$$-19b = -57$$
$$b = 3$$

$$2a + 5(3) = 7$$
$$2a + 15 = 7$$
$$2a = -8$$
$$a = -4$$

Steps for Solving Systems of Equations
(Method of Addition or Subtraction)

Step 1: Rearrange the equations so the corresponding terms line up.

Step 2: Multiply each equation by numbers that will result in the coefficients of one variable being the same.

Step 3: Subtract or add the equations to eliminate one variable.

Step 4: Solve for the remaining variable.

Step 5: Substitute this result into one of the original equations.

Step 6: Solve for the remaining variable.

Function Notation

Consider the equation of the polynomial $y = 3x^2 - 4x + 1$.

This equation, written in function notation, is $f(x) = 3x^2 - 4x + 1$.

Using this notation, you can evaluate the function at $x = -2$ or any other value of x. Evaluate $f(x)$ at $x = -2$.

$$\begin{aligned} f(-2) &= 3(-2)^2 - 4(-2) + 1 \\ &= 3(4) + 8 + 1 \\ &= 21 \end{aligned}$$

This means that when x has an assigned value of (-2), the function $f(x)$ has a value of 21. The point $(-2, 21)$ lies on the graph of $y = f(x)$, where $f(x) = 3x^2 - 4x + 1$.

Note: Other letters (either upper or lower case) can be used instead of f. Examples of letters that are often used for polynomial functions are g, h, F, P and Q.

Exercises

1. Factor the following polynomials.

   a. $x^4 + 3x^3 - 5x^2$

   b. $a^2 - 1$

   c. $y^2 + 12y + 32$

   d. $4x^2 + 3x - 10$

   e. $x^2 + 14x + 49$

2. Solve the following systems of equations.

   a. $5x - y = 42$
      $2x + 4y = 8$

   b. $a + 3b - 6 = 0$
      $4a + 8b - 20 = 0$

3. Evaluate $g(x) = 4x^3 - 3x^2 - x + 9$ for each of the given values of x.

   a. $x = 1$

   b. $x = -1$

   c. $g(2)$

   d. $g(0)$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

The Remainder Theorem

Challenge yourself

Find the remainder of $(8x^3 + 11x^2 - x + 8) + (x + 2)$ using the Remainder Theorem.

MATH FACTOR

Watch the Introduction and Video Segment 1a.

Example

If $P(x) = 2x^3 - 3x + 50$, then find the remainder when $P(x)$ is divided by $(x + 4)$ and relate the result to the graph of $y = P(x)$.

Solution

The Remainder Theorem shows that when you divide $P(x)$ by $(x + 4)$, the remainder is $P(-4)$.

$$\begin{aligned} P(-4) &= 2(-4)^3 - 3(-4) + 50 \\ &= 2(-64) - 3(-4) + 50 \\ &= -128 + 12 + 50 \\ &= -66 \end{aligned}$$

Check

Using synthetic division, divide $(2x^3 - 3x + 50)$ by $(x + 4)$.

$$\begin{array}{r|cccc} 4 & 2 & 0 & -3 & 50 \\ & & 8 & -32 & 116 \\ \hline & 2 & -8 & 29 & \boxed{-66} \end{array}$$

Therefore, when $P(x)$ is divided by $(x + 4)$, the remainder is -66.

This result means that the point $(-4, -66)$ lies on the graph of $y = P(x)$.

TEXT

Additional Examples
Mathematics 12: pp. 54-56
Mathematics: Principles and Process 12: p. 148
Holtmath 12: p. 94

Exercises

1. Write the Remainder Theorem in your own words.

2. If $P(x) = x^3 - 4x^2 + 10$, use synthetic division to find the remainder when $P(x)$ is divided by $(x - 3)$.

3. If $P(x) = x^3 - 4x^2 + 10$, then find the value of $P(3)$.

4. Use the Remainder Theorem to find the remainder of each division:

   a. $\dfrac{(x^5 + 20)}{(x + 2)}$

   b. $\dfrac{(x^3 - 2x^2 + 5x + 11)}{(x - 5)}$

5. Use the division algorithm to prove the Remainder Theorem.

EXERCISE 3a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 56 (Questions 1-6, 12, 14)
Mathematics: Principles and Process 12: p. 148 (Questions 10, 11, 13)
Holtmath 12: p. 95 (Questions 1, 2, 5, 11, 14-16)

The Factor Theorem

Challenge yourself

Complete the following questions.

1. Find the remainder when $f(x) = 2x^4 + 7x^3 + 4x^2 - 7x - 6$ is divided by each of the following:

   a. $(x - 3)$

   b. $(x + 2)$

2. From question 1, predict which binomial is a factor of $f(x)$.

MATH FACTOR

Watch Video Segment 3b.

Example 1

Determine whether $(x + 3)$ is a factor of the polynomial $f(x) = 2x^3 - 7x^2 - 24x + 45$.

Solution

$$\begin{aligned} f(-3) &= 2(-3)^3 - 7(-3)^2 - 24(-3) + 45 \\ &= 2(-27) - 7(9) + 72 + 45 \\ &= -54 - 63 + 72 + 45 \\ &= 0 \end{aligned}$$

Yes, $(x + 3)$ is a factor because the remainder is 0.

Check by dividing. You will find that $f(x) = (x + 3)(2x^2 - 13x + 15)$.

Example 2

Determine whether $(x-2)$ is a factor of the polynomial $f(x)=2x^3-7x^2-24x+45$.

Solution

$$\begin{aligned} f(2) &= 2(2)^3-7(2)^2-24(2)+45 \\ &= 2(8)-7(4)-48+45 \\ &= 16-28-48+45 \\ &= -15 \end{aligned}$$

No, $(x-2)$ is not a factor because the remainder is not 0.

Check by dividing. You will find that $f(x)=(x-2)(2x^2-3x-30)-15$.

TEXT

Additional Examples
Mathematics 12: pp. 58-61
Mathematics: Principles and Process 12: pp. 145, 146
Holtmath 12: p. 98

Exercises

6. Write the Factor Theorem in your own words. Explain how it relates to the Remainder Theorem.

7. Which of the following polynomials are divisible by $(x+2)$?

   a. x^3-3x^2-7x+6

   b. $2x^3-4x^2-7x+14$

8. Determine if each of the following binomials is a factor of $P(x) = 2x^3 - 4x^2 - 10x + 12$.

   a. $(x - 1)$

   b. $(x + 1)$

9. Use the division algorithm to prove the Factor Theorem.

EXERCISE 3b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 61 (Questions 1-21, 23, 24)
Mathematics: Principles and Process 12: p. 147 (Questions 1-3, 9)
Holtmath 12: p. 95 (Questions 5, 6, 8)

Finding Zeros and Missing Coefficients

challenge yourself

Find k if you know that $P(x) = kx^3 - 8x^2 - x + 3k + 1$ is divisible by $(x - 2)$.

MATH FACTOR

Watch Video Segment 3c.

Example

When $P(x) = 2x^3 + kx^2 + 8x - 10$ is divided by $(x - 3)$, the remainder is 5. Find k.

Solution

Using the Remainder Theorem:

$$P(3) = 5$$
$$2(3)^3 + k(3)^2 + 8(3) - 10 = 5$$
$$54 + 9k + 24 - 10 = 5$$
$$68 + 9k = 5$$
$$9k = -63$$
$$k = -7$$

Check

$$(2x^3 - 7x^2 + 8x - 10) \div (x - 3)$$

-3	2	-7	8	-10
		-6	3	-15
	2	-1	5	5

TEXT

Additional Examples
Mathematics 12: p. 56 (Question 3)

Exercises

10. When the polynomial $P(x)$ is divided by $(x + 1)$, the remainder is 3. Find the value of k if

$$P(x) = 2x^3 + 3x^2 - kx + 7$$

11. Find k if $(x+5)$ is a factor of $P(x)=x^4+5x^3-x^2+kx+35$.

12. Find the zeros of each third-degree polynomial.

a. $P(x)=x^3-4x^2-4x+16$

b. $P(x)=x^3-2x^2-3x$

EXERCISE 3c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 63 (Questions 20-22)
Mathematics: Principles and Process 12: p. 147 (Questions 6, 7)

Finding Two Missing Coefficients

Challenge yourself

If $P(x)+(x-2)$ has a remainder of 7 and $P(x)+(x+1)$ has a remainder of 10, then find the value of b in $P(x)=ax^2+bx+5$.

MATH FACTOR

Watch Video Segment 3d.

Example

When a polynomial $P(x)=x^3-ax^2+bx+9$ is divided by $(x+2)$, the remainder is 5. One factor of $P(x)$ is $(x-3)$. Find a.

Solution

Step 1
Using the Remainder Theorem:

$$P(-2)=5$$
$$(-2)^3-a(-2)^2+b(-2)+9=5$$
$$-8-4a-2b+9=5$$
$$-4a-2b=4$$

(Divide equation by -2.) $\quad 2a+b=-2$

Step 2
Using the Factor Theorem:

$$P(3)=0$$
$$(3)^3-a(3)^2+b(3)+9=0$$
$$27-9a+3b+9=0$$
$$-9a+3b=-36$$

(Divide equation by -3.) $\quad 3a-b=12$

Step 3

$$\begin{array}{r} 2a+b=-2 \\ \underline{3a-b=12\ +} \\ 5a=10 \\ a=2 \end{array}$$

Solve this system of two equations by elimination.

You could stop here because you are asked to find a, but to check, you will need to find b as well.

Step 4

$$2(2)+b=-2$$
$$4+b=-2$$
$$b=-6$$

Check by dividing $P(x)=x^3-2x^2-6x+9$ by $(x+2)$ and by $(x-3)$.

$$\begin{array}{r|cccc} 2 & 1 & -2 & -6 & 9 \\ & & 2 & -8 & 4 \\ \hline & 1 & -4 & 2 & \boxed{5} \end{array}$$

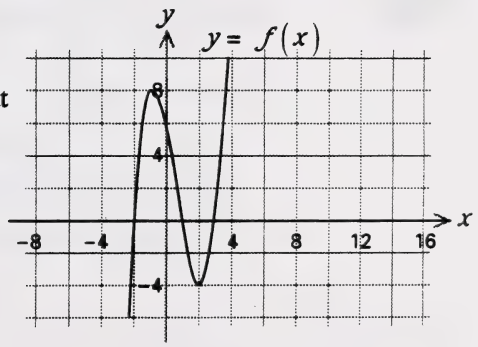
$$\begin{array}{r|cccc} -3 & 1 & -2 & -6 & 9 \\ & & -3 & -3 & 9 \\ \hline & 1 & 1 & -3 & \boxed{0} \end{array}$$

Additional Examples
Holtmath 12: p. 90

Exercises

13. When a polynomial $P(x) = x^3 + ax - bx + 6$ is divided by $(x-1)$ the remainder is -3. One factor of $P(x)$ is $(x+2)$. Find b.

14. The graph of the polynomial function $y = f(x)$, where $f(x) = x^3 - 2x^2 - 5x + 6$, is shown at the right. Use the graph to predict the factors of $f(x)$ and verify your prediction using the Factor Theorem.



15. Using the same graph of $y = f(x)$ in question 14, estimate the following and verify your results using the Remainder Theorem.

    a. the remainder of $f(x) \div (x+1)$

    b. the remainder of $f(x) \div (x-2)$

    c. the remainder of $f(x) \div x$

EXERCISE 3d SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 57 (Question 9)
Mathematics: Principles and Process 12: p. 148 (Question 12)
Holtmath 12: p. 91 (Question 5), p. 95 (Questions 3, 4, 13)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. The Factor and Remainder Theorems are used throughout the remaining lessons of this module. You should be comfortable applying them before you continue with **Lesson 4: Factoring and Graphing Polynomial Functions**.

Solutions

BUILDING BLOCKS

1. a. $x^2\left(x^2+3x-5\right)$

   b. $(a-1)(a+1)$

   c. $(y+8)(y+4)$

   d. $(4x-5)(x+2)$

   e. $(x+7)(x+7)=(x+7)^2$

2. a. $20x-4y=168$
   $2x+4y=8 \quad (\text{add})$
   $22x=176$
   $x=8$
   $2(8)+4y=8$
   $16+4y=8$
   $4y=-8$
   $y=-2$

   $x=8$
   $y=-2$

   b. $4a+12b-24=0$
   $4a+8b-20=0 \quad (\text{subtract})$
   $4b-4=0$
   $4b=4$
   $b=1$
   $a+3(1)-6=0$
   $a-3=0$
   $a=3$

   $a=3$
   $b=1$

3. a. $g(1)=4(1)^3-3(1)^2-(1)+9$
   $=9$

   b. $g(-1)=4(-1)^3-3(-1)^2-(-1)+9$
   $=3$

   c. $g(2)=4(2)^3-3(2)^2-(2)+9$
   $=27$

   d. $g(0)=4(0)^3-3(0)^2-(0)+9$
   $=9$

EXERCISE 3a

1. To find the remainder of $P(x)\div(x-a)$, evaluate $P(a)$.

2. $-3 \;\big|\; 1 \quad -4 \quad 0 \quad 10$
   $\qquad\quad -3 \quad 3 \quad 9$
   $\qquad 1 \quad -1 \quad -3 \quad \boxed{1}$

3. $P(3)=(3)^3-4(3)^2+10$
   $=27-36+10$
   $=1$

4. a. $P(-2)=(-2)^5+20$
   $=-32+20$
   $=-12$

   b. $P(5)=53-2(5)^2+5(5)+11$
   $=125-50+25+11$
   $=111$

5. See the video.

EXERCISE 3b

6. If $(x-a)$ is a factor of $P(x)$, then $P(a)=0$. This uses the Remainder Theorem and the idea that when a number is divided by a factor of that number, the remainder is zero. The Factor Theorem is a special case of the Remainder Theorem.

7. a. $P(-2)=(-2)^3-3(-2)^2-7(-2)+6$
$=-8-12+14+6$
$=0$

It is divisible by $(x+2)$.

b. $P(-2)=2(-2)^3-4(-2)^2-7(-2)+14$
$=-16-16+14+14$
$=-4$

It is not divisible by $(x+2)$.

8. a. $P(1)=2(1)^3-4(1)^2-10(1)+12$
$=2-4-10+12$
$=0$

It is a factor; thus, 1 is a zero of $P(x)$.

b. $P(-1)=2(-1)^3-4(-1)^2-10(-1)+12$
$=-2-4+10+12$
$=16$

It is not a factor; thus, -1 is not a zero of $P(x)$.

9. See the video.

EXERCISE 3c

10. $P(-1)=3$
$2(-1)^3+3(-1)^2-k(-1)+7=3$
$-2+3+k+7=3$
$8+k=3$
$k=-5$

11. $P(-5)=0$
$(-5)^4+5(-5)^3-(-5)^2+k(-5)+35=0$
$625-625-25-5k+35=0$
$-5k+10=0$
$-5k=-10$
$k=2$

12. a. $\dfrac{(x^3-4x^2-4x+16)}{(x-2)}=x^2-2x-8$

$P(x)=(x-2)(x^2-2x-8)$
$=(x-2)(x-4)(x+2)$

The zeros are 2, 4 and -2.

b. $P(x)=x(x^2-2x-3)$
$=x(x-3)(x+1)$

The zeros are 0, 3 and -1.

EXERCISE 3d

13. $P(1)=-3$
$(1)^3+a(1)^2-b(1)+6=-3$
$a-b=-10$
$P(-2)=0$
$(-2)^3+a(-2)^2-b(-2)+6=0$
$4a+2b=2$
$2a+b=1$

$a-b=-10$
$2a+b=1$
$3a=-9$
$a=-3$
$2(-3)+b=1$
$b=7$

14. The x-intercepts (zeros) are -2, 1 and 3. You can see by reading the graph that $f(-2)=0$, $f(1)=0$ and $f(3)=0$. You can verify this by evaluating the function algebraically. So, the corresponding factors are $(x+2)$, $(x-1)$ and $(x-3)$ respectively.

15. a. By reading the graph, you can see that $f(-1)$ is approximately 8. When you evaluate algebraically, you see that $f(-1)=8$. So, the remainder of $f(x)\div(x+1)$ is 8.

b. Similarly, notice that $f(2)=-4$. Thus, the remainder of $f(x)\div(x-2)$ is -4.

c. Similarly, notice that $f(0)=6$. Thus, the remainder of $f(x)\div(x-0)$ is 6.

Note: $f(x)\div x$ could also be written as $f(x)\div(x-0)$.

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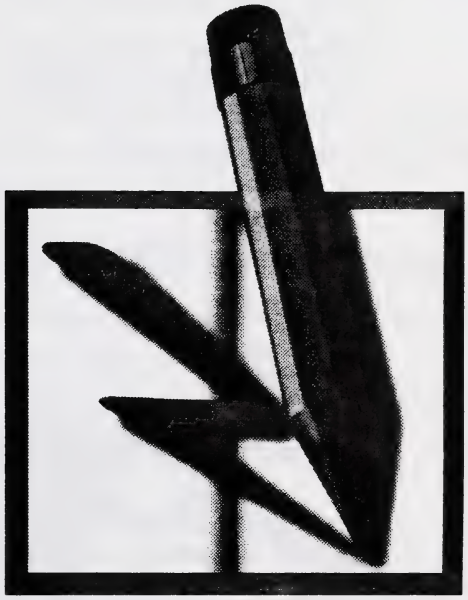
MATH FACTOR

POLYNOMIAL FUNCTIONS



Lesson 4: Factoring and Graphing Polynomial Functions

Student Workbook

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This guide accompanies **Factoring and Graphing Polynomial Functions** from the Math Factor series.

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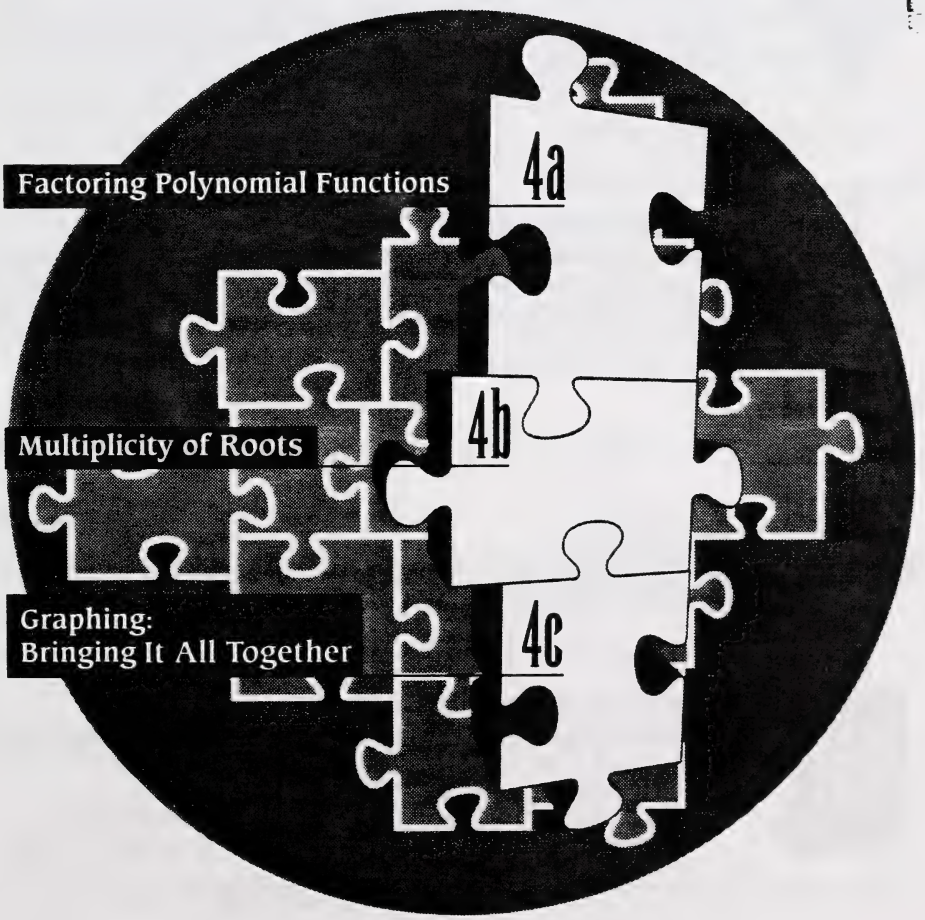
Lesson 4: FACTORING AND GRAPHING POLYNOMIAL FUNCTIONS

LEARNER EXPECTATIONS

By the end of Lesson 4 you should be able to

- factor polynomial functions using the Factor Theorem and the Rational Zero Theorem
- sketch the graphs of polynomial functions
- understand the effects of a multiplicity of roots and of the leading coefficient
- analyze the graph of a polynomial function to determine its minimum degree and estimate the roots of the equation

This lesson has three parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

The Integral Zero Theorem and Rational Zero Theorem are useful tools for factoring polynomials of degree three or higher. Before watching this program, study these theorems and do the given exercises.

Integral Zero Theorem

This theorem helps to find factors of polynomial functions. If the degree of the function is 3 or more, begin factoring by trial and error. The Integral Zero Theorem shows which integers are possible zeros.

> If a is an integral zero of the integral polynomial function $P(x)$, then a is a factor of the constant term $P(x)$.

By using this theorem, you know that the only possible integral zeros of $f(x) = x^3 - 8x^2 + 20x - 25$ are $\pm 1, \pm 5$ or ± 25. These are the only factors of 25, the constant term of the function.

Rational Zero Theorem

You may be interested in finding rational zeros as well as integral zeros. Some polynomial functions have no integral zeros. The Rational Zero Theorem shows which rational numbers are possible zeros of a polynomial function.

> If $\frac{a}{b}$ is a rational zero of the integer polynomial function $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, then a is a factor of a_0 and b is a factor of a_n.

To find the possible rational zeros of the polynomial function $P(x) = 9x^3 + 12x^2 + x - 2$, look at the factors of the first and last coefficients. The factors of 9 are 1, 3 and 9. The factors of 2 are 1 and 2. The possible rational zeros are as follows:

$$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{1}{9}, \pm\frac{2}{9}$$

Note: This function has only three rational zeros: $\frac{1}{3}, -\frac{2}{3}$ and -1.

Factoring by Grouping

Some four-term polynomials can be factored quickly by grouping. If the first two coefficients are in the same ratio as the last two coefficients, then this is possible.

$$x^3 - 3x^2 + 5x - 15 = x^2(x-3) + 5(x-3)$$
$$= (x^2 + 5)(x-3)$$

Notice the ratios of the coefficients. $1:-3 = 5:-15$

$$3x^3 - 6x^2 + 2x - 4 = 3x^2(x-2) + 2(x-2)$$
$$= (3x^2 + 2)(x-2)$$

Notice the ratios of the coefficients. $3:-6 = 2:-4$

Exercises

1. Use the Integral Zero Theorem to list the possible integral zeros of $P(x) = 3x^4 - 2x^2 + x - 4$.

2. Use the Rational Zero Theorem to list the possible rational zeros of $P(x) = 3x^4 - 2x^2 + x - 4$.

3. Using both theorems, circle the numbers that could be zeros of $Q(x) = 3x^3 - 13x^2 + 16$.

   $-1 \quad 2 \quad -3 \quad 4 \quad 13 \quad -16 \quad \frac{1}{3} \quad -\frac{2}{3} \quad -\frac{3}{2} \quad \frac{8}{3} \quad \frac{3}{8}$

4. Factor each of the following polynomials by grouping, if possible.

   a. $x^3 + 5x^2 - 2x - 10$

   b. $2x^3 - 6x^2 - 12x + 4$

   c. $x^3 + 3x^2 - 9x - 27$

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 77 (Questions 7-9, 12)
Mathematics: Principles and Process 12: p. 153 (Question 8a)
Holtmath 12: p. 99 (Questions 3, 4, 12), p. 101 (Questions 4, 11)

Factoring Polynomial Functions

Challenge yourself

Sketch the graph of $y = f(x)$, where $f(x) = 4x^3 - 11x^2 + 5x + 2$ by factoring the polynomial.

Watch the Introduction and Video Segment 4a.

Example 1

Factor $f(x) = 3x^3 - 5x^2 - 16x + 12$.

Solution

The Rational Zero Theorem shows that any rational zero would have a numerator of 1, 2, 3, 4, 6 or 12 (factors of 12) and a denominator of 1 or 3 (factors of 3).

So, possible zeros include the following:

$$\pm\frac{1}{3}, \pm\frac{2}{3}, \pm 1, \pm 1\frac{1}{3}, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

Try starting with the simplest possible zero.

$$f(1) = 3(1)^3 - 5(1)^2 - 16(1) + 12$$
$$= -6$$

Therefore, $(x-1)$ is not a factor.

$$f(-1) = 3(-1)^3 - 5(-1)^2 - 16(-1) + 12$$
$$= 20$$

Therefore, $(x+1)$ is not a factor.

$$f(2) = -16$$

Therefore, $(x-2)$ is not a factor.

$$f(-2) = 0$$

Therefore, $(x+2)$ is a factor.

Steps for Factoring Completely

Step 1: Take out common factors.
Step 2: Use the Rational Zero Theorem to choose possible zeros.
Step 3: Try these using the Factor Theorem.
Step 4: Once a factor is identified, divide.
Step 5: Repeat Steps 3 and 4 until a quadratic polynomial remains.
Step 6: Factor this using conventional means.

Divide $f(x)$ by $(x+2)$.

$$\begin{array}{r|rrrr} 2 & 3 & -5 & -16 & 12 \\ & & 6 & -22 & 12 \\ \hline & 3 & -11 & 6 & \boxed{0} \end{array}$$

$$\begin{aligned} f(x) &= (x+2)(3x^2 - 11x + 6) \\ &= (x+2)(3x-2)(x-3) \end{aligned}$$

Example 2

Sketch the graph of $y = f(x)$, where $f(x) = 3x^3 - 5x^2 - 16x + 12$.

Solution

In the previous example, $f(x)$ was factored as $f(x) = (x+2)(3x-2)(x-3)$.

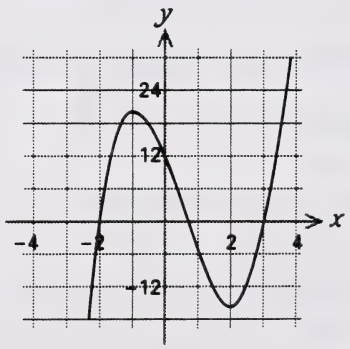
The x-intercepts of the graph are -2, $\frac{2}{3}$ and 3.

The y-intercept is $f(0) = 12$.

Steps for Sketching Graphs

Step 1: Factor the polynomial completely.
Step 2: Identify the x-intercepts using the factors.
Step 3: Identify some other point that is off the x-axis. (The y-intercept is usually the easiest to find.)
Step 4: Use the coefficient of the leading term to determine if the graph ultimately rises or falls to the right.
Step 5: Sketch the curve.

The graph will ultimately rise to the right because the first coefficient is positive.



TEXT

Additional Examples
Mathematics 12: pp. 72, 73
Mathematics: Principles and Process 12: pp. 154, 155, 157
Holtmath 12: pp. 98, 100

Exercises

1. Following the procedure given in the examples, sketch the graph of $y = f(x)$, where $f(x) = 2x^3 + 4x^2 - 10x - 12$.

EXERCISE 4a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics: Principles and Process 12: p. 153 (Questions 8-10), p. 155 (Questions 1-3), p. 158 (Questions 1-6)
Holtmath 12: p. 95 (Questions 9, 10), p. 99 (Questions 4-11), p. 101 (Questions 1-3, 5-7)

Multiplicity of Roots

Challenge yourself

Sketch the graph of $y = f(x)$, where $f(x) = x^3 - 12x - 16$.

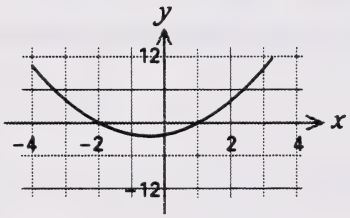
MATH FACTOR

Watch Video Segment 4b.

Example 1

Use a graphing calculator to draw the graphs $y = (x+2)^n(x-1)$ for $n = 1, 2, 3, 4, 5$. Use the same scale on your axes for each graph. The graphs show the effect of the multiplicity of a root on the shape of the graph.

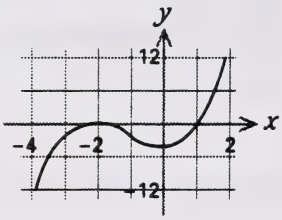
Graph 1



$y = (x+2)(x-1)$

-2 has a multiplicity of 1.

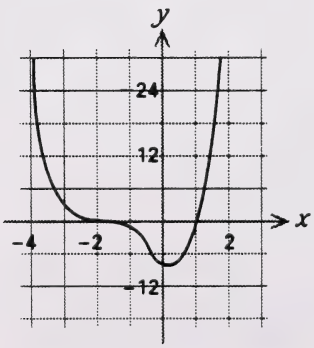
Graph 2



$y = (x+2)^2(x-1)$

-2 has a multiplicity of 2.

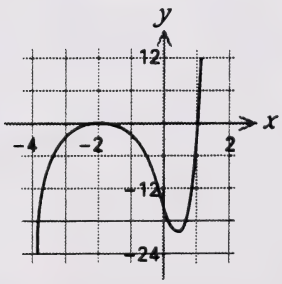
Graph 3



$y = (x+2)^3(x-1)$

-2 has a multiplicity of 3.

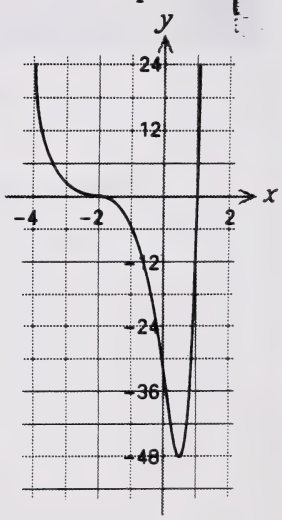
Graph 4



$y = (x+2)^4(x-1)$

-2 has a multiplicity of 4.

Graph 5



$y = (x+2)^5(x-1)$

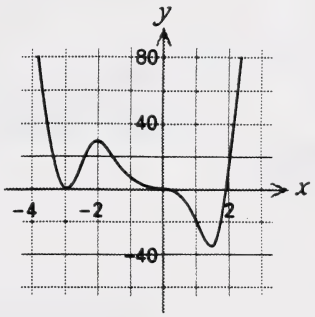
-2 has a multiplicity of 5.

Example 2

$f(x) = x^6 + 4x^5 - 3x^4 - 18x^3$ in factored form is $f(x) = x^3(x-2)(x+3)^2$. Sketch the graph of $y = f(x)$.

Solution

The roots of $f(x) = 0$ are $0, 0, 0, 2, -3$ and -3. Notice that the root of 0 has a multiplicity of three and the root of -3 has a multiplicity of two. The graph will ultimately rise to the right because the coefficient of the first term is positive. You need to find some point off the x-axis. Since the y-intercept is zero, you need another. Choose any value for x. For example, choose -2.



Since $f(-2) = 32$, $(-2, 32)$ is on the graph.

TEXT

Additional Examples
Mathematics 12: pp. 72, 73
Mathematics: Principles and Process 12: pp. 154, 155
Holtmath 12: p. 104

Exercises

2. When $f(x) = -2x^6 + 20x^5 - 78x^4 + 152x^3 - 158x^2 + 84x - 18$ is factored, the result is $f(x) = -2(x-3)^2(x-1)^4$. Sketch the graph of $y = f(x)$.

3. Factor the polynomial and sketch the graph of each of the following:

   a. $y = g(x)$, where $g(x) = x^5 + 2x^4 + x^3$

   b. $y = h(x)$, where $h(x) = x^3 - 12x^2 + 45x - 50$

EXERCISE 4b SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 76 (Questions 1-3, 5)
Mathematics: Principles and Process 12: p. 158 (Questions 10, 11)
Holtmath 12: p. 105 (Questions 1, 2, 4)

Graphing: Bringing It All Together

Challenge yourself

Sketch the graph of $y = P(x)$, where $P(x) = x^4 + 4x^3 + 3x^2 - 2x - 2$.

MATH FACTOR

Watch Video Segment 4c.

Example

Sketch the graph of $y = f(x)$, where $f(x) = x^4 - 5x^3 + 11x - 7$.

Solution

To begin factoring, try $f(1)$.

$$f(1) = (1)^4 - 5(1)^3 + 11(1) - 7$$
$$= 0$$

Therefore, $(x-1)$ is a factor.

Divide.

-1	1	-5	0	11	-7
		-1	4	4	-7
	1	-4	-4	7	0

$$f(x) = (x-1)(x^3 - 4x^2 - 4x + 7)$$

Now factor $g(x) = (x^3 - 4x^2 - 4x + 7)$.

$$g(1) = 0$$

Therefore, $(x-1)$ is a factor.

Divide.

-1	1	-4	-4	7
		-1	3	7
	1	-3	-7	0

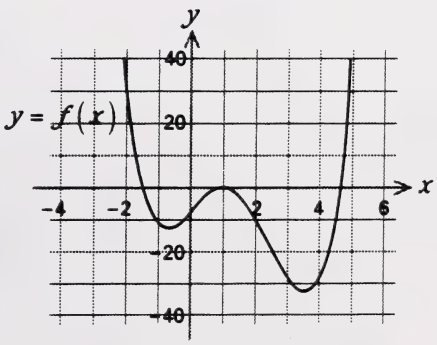
$$f(x) = (x-1)(x-1)(x^2 - 3x - 7)$$

$x^2 - 3x - 7$ cannot be factored; thus, it has no rational zeros. You can find its irrational zeros using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-7)}}{2(1)}$$
$$= \frac{3 \pm \sqrt{37}}{2}$$
$$\doteq -1.54 \text{ or } 4.54$$

The x-intercepts are -1.54, 1, 1 and 4.54.

The y-intercept is -7 because $f(0) = -7$.



TEXT

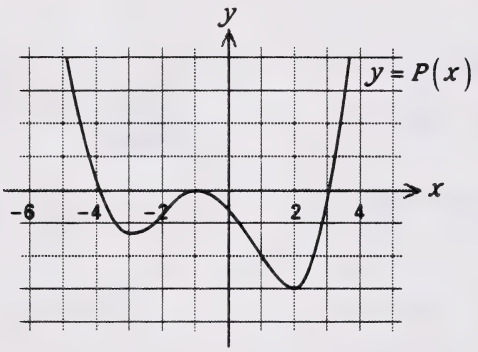
Additional Examples
Mathematics 12: pp. 72, 73
Mathematics: Principles and Process 12: pp. 154, 155
Holtmath 12: pp. 106, 109

Exercises

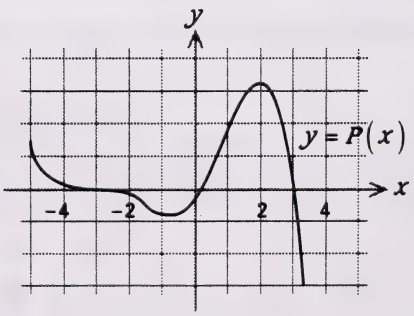
4. Identify the approximate roots of $P(x)$, and sketch the graph of $y = P(x)$, where $P(x) = 2x^3 + 7x^2 - 5$.

5. Each of the following graphs is the graph of a polynomial function. Determine the minimum possible degree of each function.

a.



b.



6. Estimate the roots of $P(x)$ for each of the functions in question 5.

a.

b.

EXERCISE 4c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 77 (Question 4), p. 78 (Questions 10, 11, 13, 14)
Holtmath 12: p. 107 (Questions 2, 5, 9), p. 109 (Questions 1-5)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. **Lesson 5: Applications and Problem Solving** will build on what you have learned.

Solutions

BUILDING BLOCKS

1. $\pm 1, \pm 2, \pm 4$

2. $\pm 1, \pm 2, \pm 4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$

3. Possible: $-1, 2, 4, -16, \frac{1}{3}, -\frac{2}{3}, \frac{8}{3}$

   Not possible: $-3, 13, -\frac{3}{2}, \frac{3}{8}$

4. a. $x^2(x+5)-2(x+5)=(x+5)(x^2-2)$

   b. It cannot be factored by grouping; in fact, it is not factorable.

   c. $x^2(x+3)-9(x+3)=(x+3)(x^2-9)$
   $=(x+3)(x+3)(x-3)$
   $=(x+3)^2(x-3)$

EXERCISE 4a

1. $f(x)=2(x^3+2x^2-5x-6)$

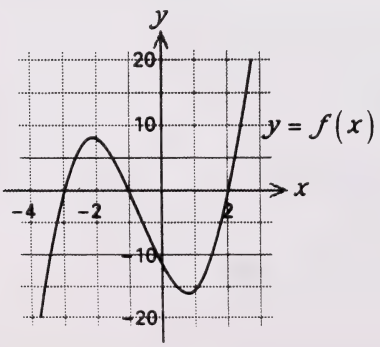
   $f(1)=-16$

   $f(-1)=0$

   $f(x)=2(x+1)(x^2+x-6)$
   $=2(x+1)(x+3)(x-2)$

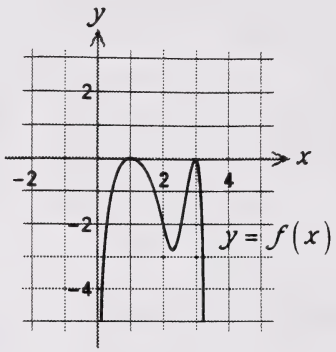
   x-intercepts are $-1, -3, 2$

   y-intercept is -12

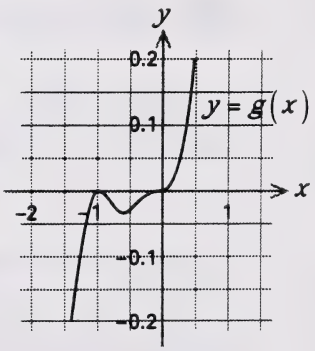


EXERCISE 4b

2.



3. a. $g(x)=x^3(x^2+2x+1)$
   $=x^3(x+1)^2$

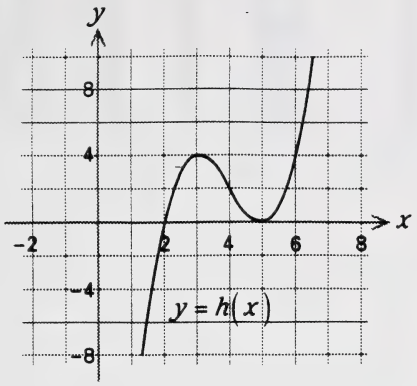


   b. $h(1)=-16$

   $h(-1)=-108$

   $h(2)=0$

   $h(x)=(x-2)(x^2-10x+25)$
   $=(x-2)(x-5)^2$



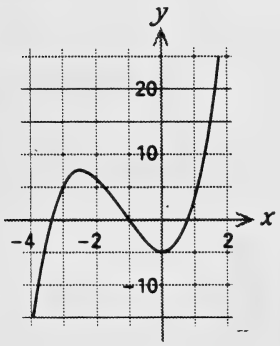
EXERCISE 4c

4. $P(1) = 4$

   $P(-1) = 0$

   $P(x) = (x+1)(2x^2 + 5x - 5)$

   The approximate roots of $2x^2 + 5x - 5 = 0$ are 0.77 and -3.27. (Use the quadratic formula.)



5. a. 4th degree
   (Count the number of intervals of increase and decrease.)

   Note: This could be 6th degree because -1 may be a root of multiplicity four; or if it has a multiplicity of six, it would be 8th degree. Judging by the shape at -1, it is probably only a double root.

   b. 5th degree
   Notice that -3 is at least a triple root.
   (Count the intervals of increase and decrease.)

   Note: This could be 7th degree because -3 may be a root of multiplicity five; or it could be 9th, 11th, 13th ... degree. Judging by the shape at -3, it is probably only a triple root.

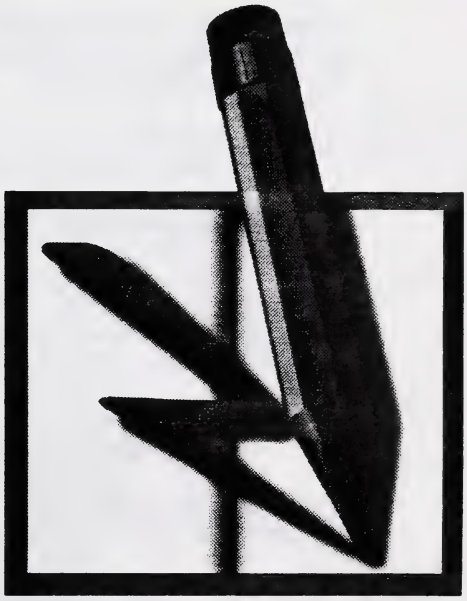
6. a. The roots are approximately -3.9, -1, -1 and 3.

   b. The roots are approximately -3, -3, -3, 0.2 and 3.

2. 1997-404 c2 US

MATH FACTOR

POLYNOMIAL FUNCTIONS



Lesson 5:
Applications and Problem Solving

Student Workbook

Produced by

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This guide accompanies **Applications and Problem Solving** from the Math Factor series.

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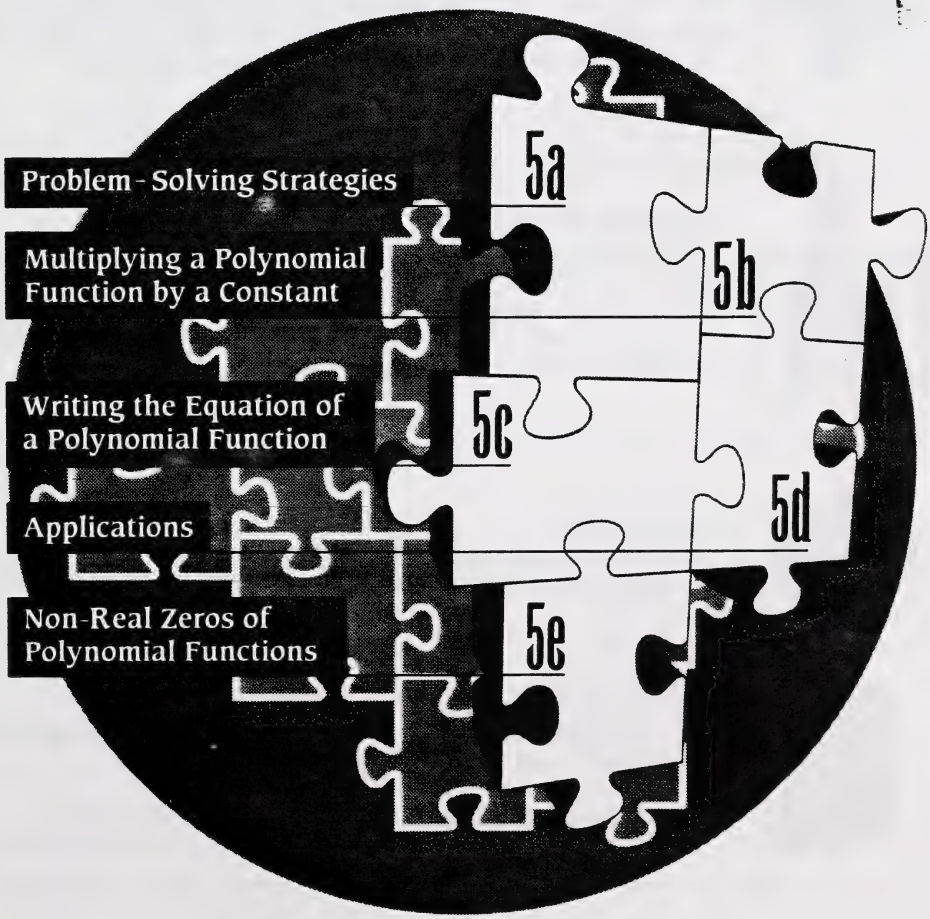
Lesson 5: APPLICATIONS AND PROBLEM SOLVING

LEARNER EXPECTATIONS

By the end of Lesson 5 you should be able to

- understand the effects of multiplying an entire polynomial function by a constant
- write the equation of a polynomial function given the roots of the equation and a point other than the x-intercepts, on the graph of the function.
- solve problems that can be represented by polynomial functions such as problems involving optimum dimensions of containers and involving integers
- recognize that some polynomial functions have non-real roots

This lesson has five parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

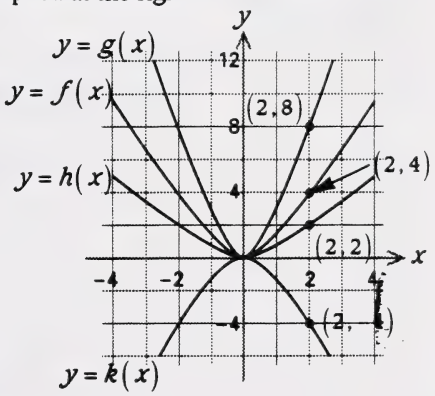
- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Building Blocks

Do you remember the effects of multiplying a function by a constant? Before watching this program, refresh your memory by studying the following examples and doing the exercises.

Vertical Stretch and Reflection

For example, $y = f(x)$, where $f(x) = x^2$, is graphed at the right.



Also graphed are the following:

- $y = g(x)$, where $g(x) = 2f(x)$
- $y = h(x)$, where $h(x) = \frac{1}{2}f(x)$
- $y = k(x)$, where $k(x) = -f(x)$

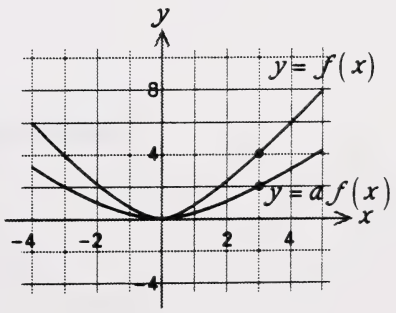
When you multiply a function by a constant, you are actually multiplying the y-coordinate of each point on the graph by this number.

- Notice that $(2, 4)$ is on $y = f(x)$.

  Then $(2, 8)$ is on $y = g(x)$, where $g(x) = 2f(x)$, $(2, 2)$ is on $y = h(x)$, where $h(x) = \frac{1}{2}f(x)$, and $(2, -4)$ is on $y = k(x)$, where $k(x) = -f(x)$.

Exercises

1. Given the graphs of $y = f(x)$ and $y = af(x)$ at the right, find the value of a.



2. If $f(x) = x^2 - 4x + 2$, write an expression for $y = -2f(x)$.

3. If the point $(-2, -8)$ lies on the graph of $y = f(x)$, then what would be the corresponding point on the graph of $y = -\frac{1}{4}f(x)$?

BUILDING BLOCKS SOLUTIONS are at the back of this workbook.

Problem-Solving Strategies

challenge yourself

Describe key elements of successful problem solving.

Watch the Introduction and Video Segment 5a.

Exercises

1. Outline three steps for successful problem solving.

EXERCISE 5a SOLUTIONS are at the back of this workbook.

Multiplying a Polynomial Function by a Constant

Challenge yourself

Predict the effects of the different values of the parameter a on the graph of $y = P(x)$, where $P(x) = a(x^3 + 6x^2 + 9x + 4)$.

MATH FACTOR

Watch Video Segment 5b.

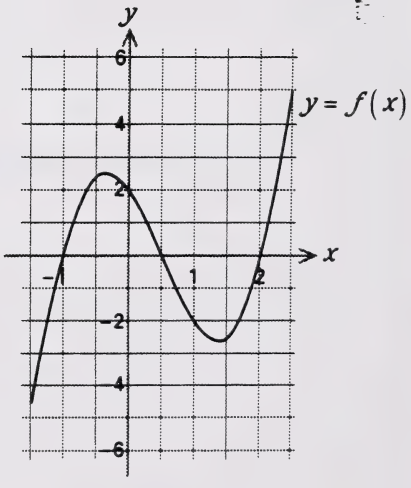
TEXT

Additional Examples
Mathematics 12: pp. 74, 75

Exercises

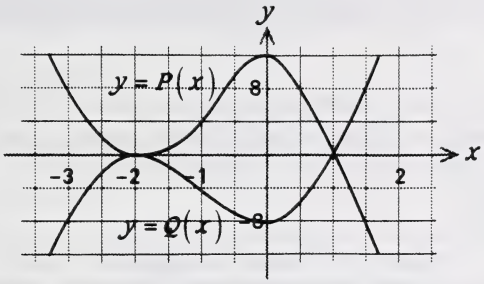
2. Given the graph of $y = f(x)$, where $f(x) = 2x^3 - 3x^2 - 3x + 2$, sketch the graphs of the following:

   - $y = g(x)$, where $g(x) = 2f(x)$
   - $y = h(x)$, where $h(x) = \frac{1}{2}f(x)$
   - $y = k(x)$, where $k(x) = -2f(x)$



EXCELLENCE

3. The following are the graphs of $y = P(x)$ and $y = Q(x)$, where $Q(x) = 2a \times P(x)$. Find the value of a.



4. When a polynomial $P(x)$ is divided by $(x-4)$, the remainder is -2. If $3P(x)$ is divided by $(x-4)$, what will be the remainder? Explain.

EXERCISE 5b SOLUTIONS are at the back of this workbook.

Writing the Equation of a Polynomial Function

Challenge yourself

Write the equation of the third-degree polynomial function whose graph passes through the x-axis at $-\frac{10}{3}$, touches the x-axis at the origin, and passes through the point $(-2, -8)$.

MATH FACTOR

Watch Video Segment 5c.

Example

Write the equation of the third-degree polynomial function where the graph passes through the x-axis at 2, touches the x-axis at -1, and passes through the y-axis at 40.

It is important that the question states that it is a third-degree polynomial. Otherwise, "touching" the x-axis could indicate a multiplicity of two, four, six, eight and so on.

Solution

The zeros of the function will be 2 and -1. Because the graph touches the x-axis at -1, the root -1 has a multiplicity of two. Therefore, the factors are $(x-2)(x+1)^2$.

$$f(x) = k(x-2)(x+1)^2$$

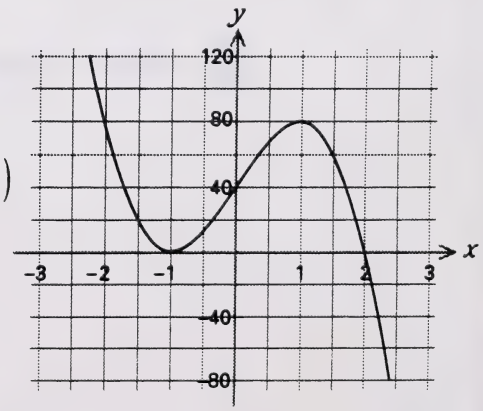
$f(0) = 40$ (i.e., the y-intercept)

$$40 = k(0-2)(0+1)^2$$
$$40 = k(-2)(1)$$
$$40 = -2k$$
$$k = -20$$

To find the constant k, substitute into the equation the values of a point on the graph.

$$\therefore f(x) = -20(x-2)(x+1)^2$$
$$= -20(x-2)(x^2+2x+1)$$
$$= -20(x^3+2x^2+x-2x^2-4x-2)$$
$$= -20(x^3-3x-2)$$
$$= -20x^3+60x+40$$

Check by graphing on a graphing calculator or graphing software.



TEXT

Additional Examples
Mathematics 12: p. 75
Mathematics: Principles and Process 12: p. 156
Holtmath 12: p. 106

Exercises

5. Write the equation of the fourth-degree polynomial function where the graph has a y-intercept of 90, passes through the x-axis at $2\frac{1}{2}$ and at -5, and touches the x-axis at -3.

6. a. Write the equation of the third-degree polynomial function where the graph passes through the points $(0, 0)$ and $(1, 36)$ and touches the x-axis at $(3, 0)$.

   b. Write the equations of two other third-degree polynomial functions with the same zeros as in question 6a.

7. Describe what information is necessary in order for you to write the equation of a unique polynomial function.

EXERCISE 5c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 77 (Questions 1, 6)
Mathematics: Principles and Process 12: p. 156 (Questions 6-8)
Holtmath 12: p. 101 (Questions 8, 9), p. 105 (Question 3), p. 107 (Questions 3, 6)

Applications

Challenge yourself

Find the optimal dimensions of an open box, cut out of a piece of cardboard 20 cm by 30 cm, which would hold 1000 cm^3.

MATH FACTOR

Watch Video Segment 5d.

Example

Find three consecutive odd integers that have a product of 693.

Solution

Let x be the lowest of these integers. Then the other two numbers are $(x+2)$ and $(x+4)$.

$$\text{Product} = x(x+2)(x+4)$$
$$693 = x(x^2+6x+8)$$
$$693 = x^3+6x^2+8x$$
$$0 = x^3+6x^2+8x-693$$

The possible zeros are as follows:
$\pm 1, \pm 3, \pm 7, \pm 9, \pm 11, \pm 21 \ldots$

$P(1) = -678$
$P(-1) = -696$
$P(3) = -588$
$P(-3) = -690$
$P(7) = 0$
Therefore, $(x-7)$ is a factor.

Divide.

$$\begin{array}{r|rrrr} -7 & 1 & 6 & 8 & -693 \\ & & -7 & -91 & -693 \\ \hline & 1 & 13 & 99 & \boxed{0} \end{array}$$

$$0 = (x-7)(x^2+13x+99)$$

$0 = x-7$ or $0 = x^2+13x+99$
$x = 7$ no real roots

The other numbers are $(x+2)$ and $(x+4)$; thus, the numbers are 7, 9 and 11.

TEXT

Additional Examples
Mathematics 12: pp. 64-66

Exercises

8. Find four consecutive integers that have a product of 1680.

9. A rectangular piece of cardboard measures 24 cm by 16 cm. A square is cut out of each corner so that the cardboard can be folded into a box with a volume of 480 cm^3. What could the dimensions of the box be?

EXERCISE 5d SOLUTIONS are at the back of this workbook.

TEXT

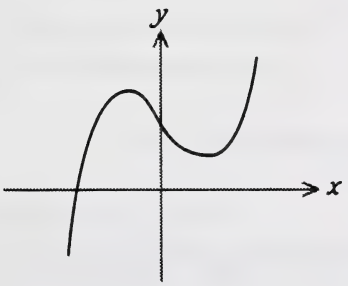
Additional Practice
Mathematics 12: p. 66 (Questions 1-9)
Mathematics: Principles and Process 12: p. 159 (Questions 12-16, 18, 19)
Holtmath 12: p. 111 (Questions 1-3)

Non-Real Zeros of Polynomial Functions

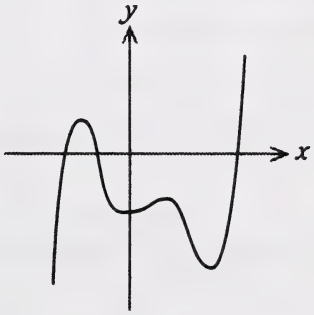
MATH FACTOR

Watch Video Segment 5e.

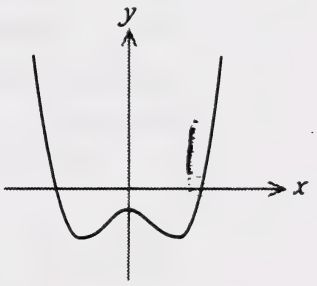
The degree of a polynomial function is the maximum number of real zeros. Some functions have less than this number of real zeros. The remaining zeros are non-real and they always occur in pairs. Examine the following sketches of the graphs of polynomial functions. Each of the functions has some non-real zeros.



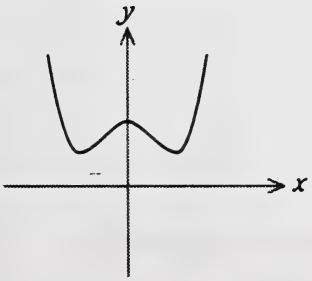
degree 3
1 real zero
2 non-real zeros



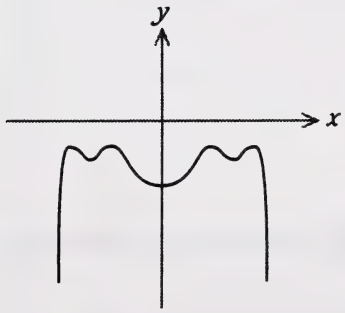
degree 5
3 real zeros
2 non-real zeros



degree 4
2 real zeros
2 non-real zeros



degree 4
4 non-real zeros



degree 8
8 non-real zeros

TEXT

Additional Examples
Holtmath 12: p. 102

Exercises

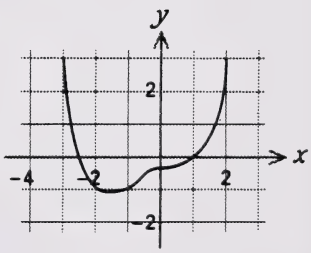
10. For the cubic function $f(x) = x^3 + 2x^2 + x - 1$, do the following:

    a. Make a table of values and sketch the graph. Choose values of x between -3 and 2, with increments of 0.5.

    b. Estimate the zeros of the function from your graph.

    c. How many non-real zeros does this function have?

11. $y = P(x)$, where $P(x) = 2x^4 + 3x^3 - 3x^2 + 3x - 5$ is sketched at the right. How many non-real zeros does $P(x)$ have?



EXERCISE 5e SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 78 (Question 14)
Holtmath 12: p. 102 (Questions 1-3), p. 107 (Question 1)

WHAT HAVE YOU LEARNED?
Return to the Learner Expectations at the beginning of the lesson and check off the points that you have mastered. You should be comfortable with solving polynomial function problems before you continue with **Lesson 6: Review**.

Solutions

BUILDING BLOCKS

1. What is multiplied by 4 to get 2?

$$4a = 2$$
$$a = \frac{1}{2}$$

2. Multiply each term by -2.

$$y = -2x^2 + 8x - 4$$

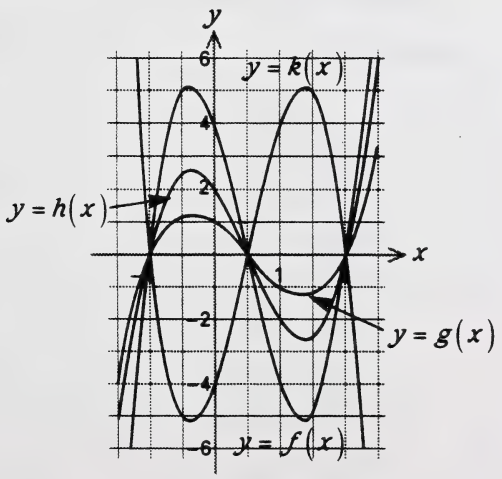
3. The y-value of the corresponding point would be $-8 \times \left(-\frac{1}{4}\right) = 2$. The coordinates are $(-2, 2)$.

EXERCISE 5a

1. The three major steps are as follows:
   - Understand the problem.
   - Develop and implement a plan.
   - Look back to see that you have solved the problem properly.

EXERCISE 5b

2.



3. $Q(x) = 2a \times P(x)$
$Q(0) = 2a \times P(0)$
$-8 = 2a \times P(12)$
$-8 = 24a$
$a = -\frac{1}{3}$

4. The remainder is -6. Using the Remainder Theorem, you know that $P(4) = -2$ and that $(4, -2)$ lies on the graph of $P(x)$. The graph of $3P(x)$ is a vertical stretch by a factor of 3; therefore, $(4, -6)$ lies on the $3P(x)$. So, the remainder is -6.

EXERCISE 5c

5. zeros $= \frac{5}{2}, -5$ and -3
The zero of -3 has a multiplicity of two.

$$f(x) = k(2x-5)(x+5)(x+3)^2$$
$$f(0) = k[2(0)-5](0-5)(0+3)^2$$
$$90 = k[2(0)-5](0-5)(0+3)^2$$
$$90 = k(-5)(-5)(9)$$
$$90 = -225k$$
$$k = -0.4$$

$$\therefore f(x) = -0.4(2x-5)(x+5)(x+3)^2$$
$$= -0.4(2x^4 + 17x^3 + 23x^2 - 105x - 225)$$
$$= -0.8x^4 - 6.8x^3 - 9.2x^2 + 42x + 90$$

6. a. zeros = 3 and 0
The root of 3 has a multiplicity of two.

$$f(x) = kx(x-3)^2$$
$$f(1) = k(1)(1-3)^2$$
$$36 = k(1)(1-3)^2$$
$$36 = k(1)(4)$$
$$36 = 4k$$
$$k = 9$$

$$\therefore f(x) = 9x(x-3)^2$$
$$= 9x^3 - 54x^2 + 81x$$

b. Multiply the equation by any constant. For example,

$$y = 18x^3 - 108x^2 + 161x$$
$$y = 90x^3 - 540x^2 + 810x$$

7. To write the equation, you need all of the zeros of the function and any other point on the graph of the function.

EXERCISE 5d

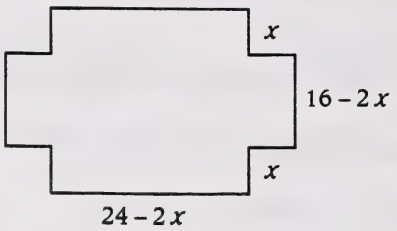
8. Let x be the lowest integer. Thus, the others are $(x+1), (x+2)$ and $(x+3)$.

$$1680 = x(x+1)(x+2)(x+3)$$
$$1680 = x^4 + 6x^3 + 11x^2 + 6x$$
$$0 = x^4 + 6x^3 + 11x^2 + 6x - 1680$$
$$0 = (x-5)(x^3 + 11x^2 + 66x + 336)$$
$$0 = (x-5)(x+8)(x^2 + 3x + 42)$$

$x^2 + 3x + 42$ has no real zeros.
$x = 5, -8$

Since the lowest number could be either 5 or -8, the integers could be either 5, 6, 7 and 8 or -8, -7, -6 and -5. You could have found the roots by graphing the polynomial. The problem with graphing is that one can only estimate the roots, but since the question indicates that the solutions are integers, the problem is inconsequential.

9.



$$\text{Volume} = x(16-2x)(24-2x)$$
$$480 = x(16-2x)(24-2x)$$
$$480 = x(384 - 80x + 4x^2)$$
$$480 = 384x - 80x^2 + 4x^3$$
$$0 = 4x^3 - 80x^2 + 384x - 480$$
$$0 = x^3 - 20x^2 + 96x - 120$$
$$0 = (x-2)(x^2 - 18x + 60)$$

$x = 2, 13.58, 4.42$

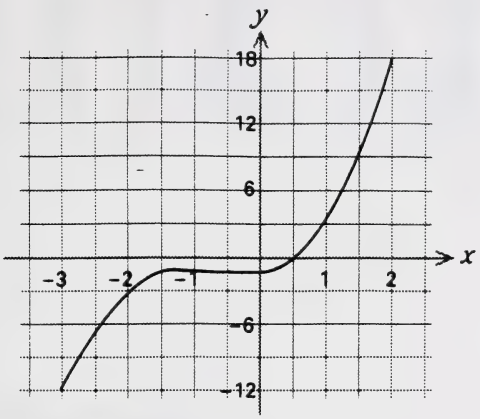
If $x = 2$, then the dimensions are 2, 12 and 20.
If $x = 13.48$, then the dimensions are 13.48, -11.56 and -3.56 (which is impossible).
If $x = 4.42$, then the dimensions are 4.42, 7.17 and 15.17.

Therefore, the dimensions are either 2 by 12 by 20 cm or approximately 4.42 by 7.17 by 15.17 cm.

EXERCISE 5e

10. a.

x	$f(x)$
−3.0	−13.0
−2.5	−6.6
−2.0	−3.0
−1.5	−1.4
−1.0	−1.0
−0.5	−1.1
0.0	−1.0
0.5	0.1
1.0	3.0
1.5	8.4
2.0	17.0



b. Approx. zero = 0.5

c. This function has two non-real zeros and one real zero.

11. $P(x)$ has two non-real zeros and two real zeros.

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MATH FACTOR

POLYNOMIAL FUNCTIONS

Lesson 6: Review

Student Workbook

Produced by

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for

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This guide accompanies **Review** from the Math Factor series.

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Producer: Katherine Rankin

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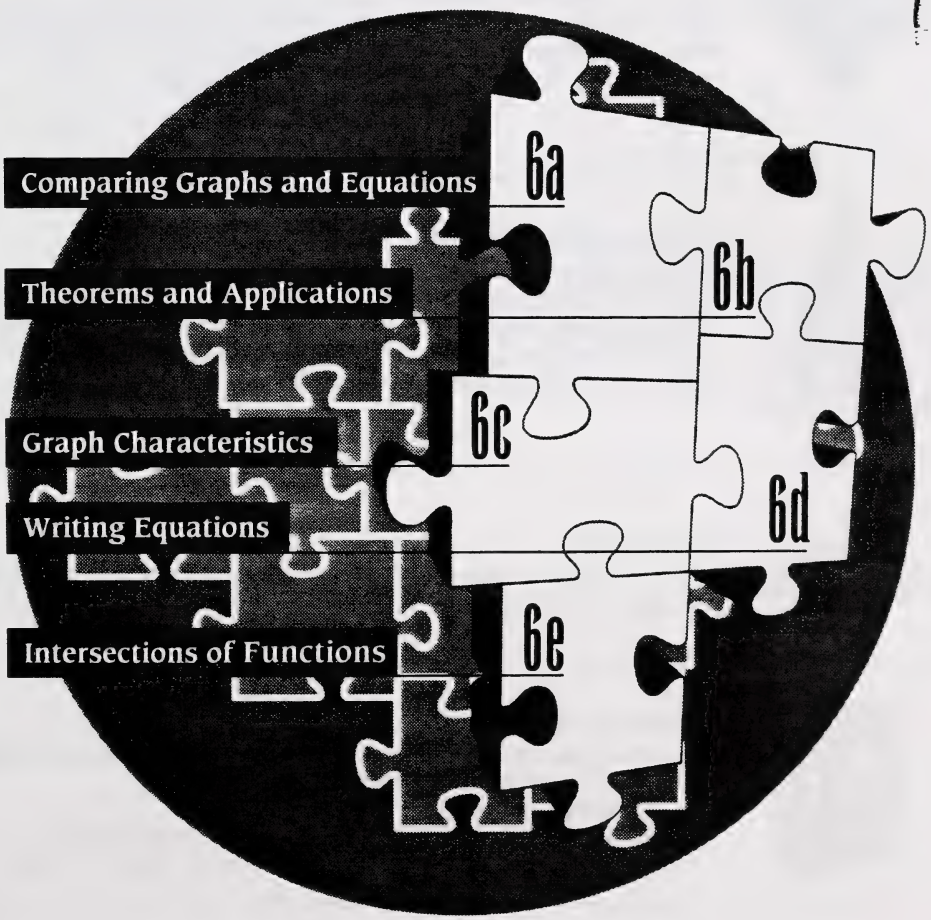
Lesson 6: REVIEW

LEARNER EXPECTATIONS

By the end of Lesson 6 you should be able to

- analyze problems in-depth linking various concepts from the module
- examine the variety of approaches to the solutions of individual problems
- use and understand the language of mathematics as it relates to polynomial functions
- solve problems involving the intersection of two polynomial functions

This lesson has five parts:



EXPLANATION OF ICONS

Building Blocks:

Do you remember most of your Mathematics 20? This course builds upon many of those skills and concepts. The Building Blocks icon appears in the workbook to indicate review examples and/or exercises. To refresh your memory, do this section in the workbook before watching the video.

Video:

Whenever this icon appears in the workbook, you should watch the indicated segment of the video program. Preceding this icon, there is often a problem titled "Challenge yourself." Try the problem before watching the video. As you work through a program segment, stop the video to

- challenge yourself to find the solution before it is given
- write down key ideas and examples
- reflect on what is being taught

Challenge Yourself:

This icon sometimes appears on the television screen when a new example is starting. When it appears, it is suggested that you stop the video and try the example yourself before proceeding. For your convenience, sometimes the example is already written in the workbook. The solution will be given in the video.

Student Workbook:

This icon appears on the television screen at the end of teaching segments, when a new skill or sub-skill has been taught. The icon indicates that the workbook contains related examples and/or exercises. Stop the video to study the examples and complete the exercises. The video icon in the workbook will indicate when you should return to the video and move on to learning a new skill.

Excellence:

This icon appears in the workbook to indicate advanced level examples or exercises that you might find challenging. Test your new skills!

Textbook Reference:

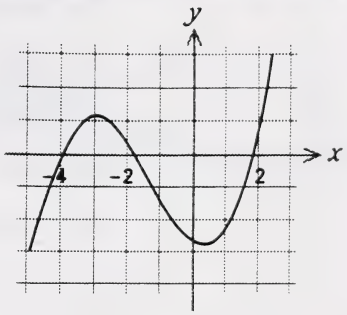
This icon appears in the workbook to indicate additional examples and exercises that can be found in the following three textbooks:

- *Mathematics 12 (Addison-Wesley)*
- *Mathematics: Principles and Process 12 (Nelson Canada)*
- *Holtmath 12 (Holt, Rinehart and Winston of Canada, Limited)*

Comparing Graphs and Equations

Challenge yourself

Verify that the graph given on the right could represent the equation $y = 4x^3 + 16x^2 - 12x - 48$.

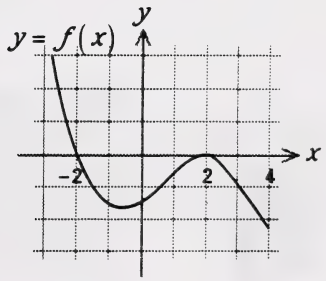


MATH FACTOR

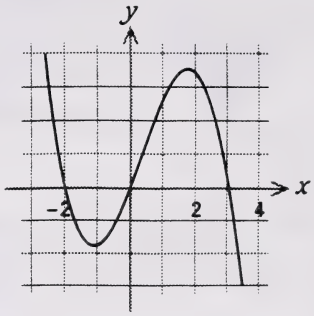
Watch the Introduction and Video Segment 6a.

Exercises

1. Verify that the given graph could represent the graph of $y = f(x)$, where
$f(x) = -3x^3 + 6x^2 + 12x - 24$.



2. Decide which equation could represent the graph of $y = P(x)$ shown below. Explain how you know that your choice is correct.



A. $P(x) = -5(x-3)(x+2)$

B. $P(x) = 5(x+3)(x-2)$

C. $P(x) = -5x(x-3)(x+2)$

D. $P(x) = 5x(x-3)(x+2)$

E. $P(x) = 5x(x+3)(x-2)$

EXERCISE 6a SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 84 (Questions 9, 10)
Mathematics: Principles and Process 12: p. 169 (Question 5)
Holtmath 12: p. 113 (Questions 6, 9, 11, 13), p. 114 (Questions 6, 8, 10)

Theorems and Applications

Watch Video Segment 6b.

Example 1

Use synthetic division to determine the remainder when $P(x)=\left(x^3+3x^2-13x+23\right)$ is divided by $(x-2)$. Verify your result using the division algorithm.

Solution

$$\begin{array}{r|rrrr} -2 & 1 & 3 & -13 & 23 \\ & & -2 & -10 & 6 \\ \hline & 1 & 5 & -3 & \boxed{17} \end{array}$$

The remainder is 17.

$$\begin{aligned} P(x) &= Q(x)D(x)+R(x) \\ &= \left(x^2+5x-3\right)(x-2)+17 \\ &= x^3+3x^2-13x+6+17 \\ &= x^3+3x^2-13x+23 \end{aligned}$$

The Remainder Theorem could be used to find the remainder. This theorem is good for verification of solutions.

Example 2

$(x-3)$ is a factor of $P(x)=2x^3-hx^2+kx+6$. When $P(x)$ is divided by $(x+3)$, the remainder is -114. Find k.

Solution

Since $(x-3)$ is a factor, $P(3)=0$.

Use the Factor Theorem.

$$\begin{aligned} 0 &= 2(3)^3-h(3)^2+k(3)+6 \\ &= 54-9h+3k+6 \\ &= -9h+3k+60 \end{aligned}$$

Since $\frac{P(x)}{(x+3)}$ has a remainder of -114, then $P(-3)=-114$.

$-114 = 2(-3)^3 - h(-3)^2 + k(-3) + 6$ Use the Remainder Theorem.

$-114 = -54 - 9h - 3k + 6$

$0 = -9h - 3k + 66$

$0 = -9h - 3k + 60$

$0 = -9h - 3k + 66$ (subtract) Solve a system of two linear equations.

$0 = 6k - 6$

$6 = 6k$

$k = 1$

Exercises

3. If $P(x) = (x+3)Q(x) + R$, then find $P(-3)$.

4. When $x^5 + mx - 2$ is divided by $x - 1$, the remainder is 7. Find the value of m.

5. The polynomial $P(x) = (x+2)(x^2 + 2x - 1) + k$ is divisible by $(x+4)$. Find the value of k.

6. Two factors of the polynomial $2x^3 - hx + k$ are $(x+2)$ and $(x-1)$. Find the value of h.

7. The polynomial $P(x) = 3x^4 - 40$ is divided by $(x - 2)$. Find the coefficient of the x^2 term in the quotient.

8. Two of the zeros of a third-degree polynomial of the form $P(x) = ax^3 + bx + c$ are 2 and -1. If $P(0) = 6$, then do the following.

   a. Find the polynomial.

   b. Find the third zero.

   c. Sketch the graph of $y = P(x)$.

EXERCISE 6b SOLUTIONS are at the back of this workbook.

Additional Practice
Mathematics 12: p. 84 (Questions 2-7, 11)
Mathematics: Principles and Process 12: p. 169 (Questions 1-3), p. 170 (Questions 1-3)
Holtmath 12: p. 113 (Questions 2, 3), p. 114 (Questions 2, 3, 5, 16)

Graph Characteristics

MATH FACTOR

Watch Video Segment 6c.

Example

What can you predict about the graph of $y = P(x)$, where $P(x) = 6x^4 - 39x^3 + 84x^2 - 60x$?

Solution

The characteristics of the graph are as follows:

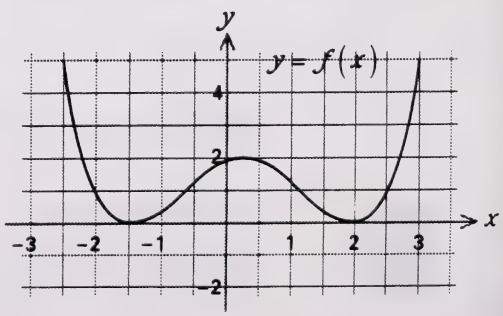
- The graph rises to the right because the initial coefficient is positive $(+6)$.
- The y-intercept is zero because there is no constant term. The constant term gives the y-intercept, and in $P(x)$ it is zero.
- It could have zero, one, two, three or four x-intercepts because it is a fourth-degree polynomial. If it were an odd-degree polynomial, its number of x-intercepts would be between one and the number of the degree.
- Using the Integral Zero Theorem, the integral zeros must be factors of 60 if there are any.
- Using the Rational Zero Theorem, the rational zeros must have a numerator that is a factor of 60 and a denominator that is a factor of 6 if there are any.
- You can find the exact zeros by factoring $P(x)$. Since $P(x) = 3x(x-2)^2(2x-5)$, the zeros are 0, 2 and $\frac{5}{2}$.

  This also means that the graph of $y = P(x)$ has x-intercepts of 0, 2 and $\frac{5}{2}$ and that the roots of the equation $P(x) = 0$ are 0, 2 and $\frac{5}{2}$. The root 2 has a multiplicity of two.

Exercises

9. The graph of the polynomial function $y = f(x)$ is shown at the right.

   a. Determine the minimum possible degree of the function.



b. Determine the multiplicity of the largest root of the equation $f(x) = 0$.

c. Determine an estimate of the constant term of $f(x)$.

d. Determine at least three factors of $f(x)$.

e. Determine an equation of $f(x)$, assuming it is degree 4.

EXERCISE 6c SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 84 (Question 10)
Mathematics: Principles and Process 12: p. 169 (Question 6), p. 170 (Question 5)
Holtmath 12: p. 113 (Questions 1, 7, 14, 15), p. 114 (Questions 4, 12, 14, 15)

Writing Equations

Challenge yourself

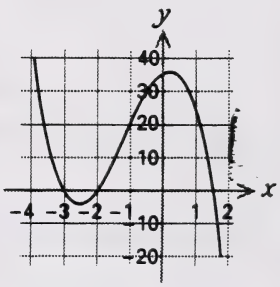
Sketch the graph of and write the equation of a cubic polynomial function $P(x)$ with zeros -2 and 3 where $P(0) = 12$.

MATH FACTOR

Watch Video Segment 6d.

Exercises

10. The graph of a third-degree polynomial function with a y-intercept of 36 is shown. Find the equation of the function that describes the graph. Clearly show all steps you take to determine the equation.



11. The graph of a third-degree polynomial function passes through the x-axis at -3 and touches it at -2. The graph passes through the point $Q(1, 144)$. Find the y-intercept of this graph.

12. The graph of a third-degree polynomial function has x-intercepts of 2 and -1 and a y-intercept of -4.

    a. Sketch all the possible graphs.

    b. Find the equations of all the possible functions that satisfy the given conditions.

EXERCISE 6d SOLUTIONS are at the back of this workbook.

TEXT

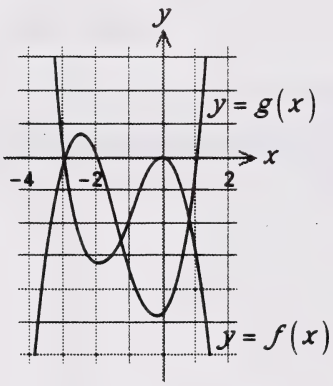
Additional Practice
Mathematics 12: p. 77 (Question 6)
Mathematics: Principles and Process 12: p. 169 (Question 4), p. 170 (Question 8)
Holtmath 12: p. 105 (Question 3)

Intersections of Functions

There is no video segment for 6e: Intersections of Functions.

Example

The graphs of the polynomial functions $y = f(x)$, where $f(x) = -x^3 - 3x^2$ and $y = g(x)$, where $g(x) = x^3 + 4x^2 + x - 6$ are shown below. Find the points of intersection of these two graphs.



Solution

$$f(x) = g(x)$$
$$-x^3 - 3x^2 = x^3 + 4x^2 + x - 6$$
$$0 = 2x^3 + 7x^2 + x - 6$$

Solve by factoring.

You know that $x = -3$ is one solution because the graphs intersect there. Therefore, $(x + 3)$ is a factor.

3	2	7	1	−6
		6	3	−6
	2	1	−2	0

$$0 = (x + 3)(2x^2 + x - 2)$$
$$0 = 2x^2 + x - 2 \quad \text{or} \quad x + 3 = 0$$

Finding the Intersections of Two Graphs $y = f(x)$ and $y = g(x)$

Step 1: Let $f(x) = g(x)$.
Step 2: Solve the resulting equation for x.
Step 3: Substitute these x-values into either $f(x)$ or $g(x)$ to find the corresponding y-coordinates.

Use the quadratic formula to find the other factors.

$$x = \frac{-1 \pm \sqrt{1^2 - 4(2)(-2)}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{17}}{4}$$

$$x \doteq -3, -1.28, 0.78$$

To find the coordinates of the points of intersection, substitute into $f(x)$ or $g(x)$.

$f(-3) = 0$

$f(-1.28) \doteq -2.82$

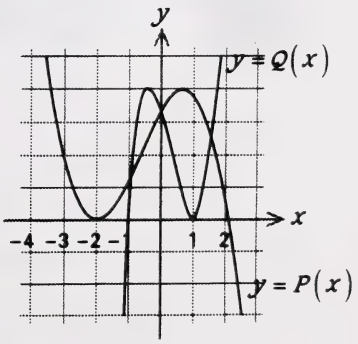
$f(0.78) \doteq -2.30$

Therefore, the three points of intersection of $f(x)$ and $g(x)$ are approximately $(-3, 0)$, $(-1.28, -2.82)$ and $(0.78, -2.30)$.

Exercises

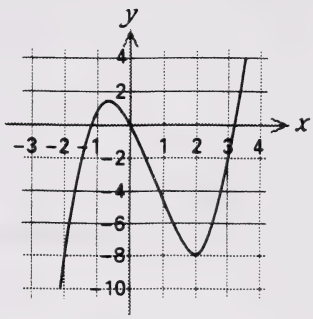
EXCELLENCE

13. The graphs of $y = P(x)$, where $P(x) = -x^3 - 2x^2 + 4x + 8$, and $y = Q(x)$, where $Q(x) = 8x^3 - 8x^2 - 8x + 8$, are given below. Find the points of intersection of $y = P(x)$ and $y = Q(x)$.



EXCELLENCE

14. The graph of $y = F(x)$, where $F(x) = x^3 - 2x^2 - 4x$ is given below. What values of x satisfy $F(x) = -8$? Explain what this means in terms of the graph of $y = F(x)$.



15. For the graph in question 14, approximate the value of the x-intercepts.

EXERCISE 6e SOLUTIONS are at the back of this workbook.

TEXT

Additional Practice
Mathematics 12: p. 84 (Question 8)
Holtmath 12: p. 113 (Questions 5, 10)

Solutions

EXERCISE 6a

1. $f(x) = -3(x^3 - 2x^2 - 4x + 8)$
$= -3[x^2(x-2) - 4(x-2)]$
$= -3(x^2 - 4)(x-2)$
$= -3(x-2)(x+2)(x-2)$
$= -3(x-2)^2(x+2)$

From the factored form, there is a root of -2 and a root with multiplicity two at 2. So, the graph passes through the x-axis at -2 and touches it at 2. You also see that the y-intercept should be -24 because the constant term is -24. Because the y-intercept of the graph shown is negative and there is no scale given, this graph could represent $y = f(x)$.

2. Since the graph passes through the x-axis at -2, 0 and 3, you know that $(x+2)$, x and $(x-3)$ are factors of the function. Equations C and D have these factors. To determine which is the correct equation, observe how the graph falls to the right. Because of this, you know that the initial coefficient should be negative. Therefore, Equation C is correct.

EXERCISE 6b

3. $P(-3) = (-3+3)Q(-3) + R$
$= 0 + R$
$= R$

4. $P(x) = x^5 + mx - 2$
$P(1) = 7$ using the Remainder Theorem
$\therefore 7 = (1)^5 + m(1) - 2$
$7 = 1 + m - 2$
$7 = m - 1$
$m = 8$

5. $P(-4) = 0$
$\therefore 0 = [(-4)+2][(-4)^2 + 2(-4) - 1] + k$
$0 = (-2)(7) + k$
$0 = -14 + k$
$k = 14$

6. $P(-2) = 0$
$\therefore 0 = 2(-2)^3 - h(-2) + k$
$0 = -16 + 2h + k$
$P(1) = 0$
$\therefore 0 = 2(1)^3 - h(1) + k$
$0 = 2 - h + k$

$0 = -16 + 2h + k$
$0 = 2 - h + k$ (subtract)
$0 = -18 + 3h$
$h = 6$

7.

2	3	0	0	0	−40
		6	−12	24	−48
	3	−6	12	−24	8

The quotient is $3x^3 - 6x^2 + 12x - 24$.
The x^2 coefficient is -6.

8. a. $P(x) = ax^3 + bx + c$
$P(0) = 6$
$\therefore c = 6$
$P(x) = ax^3 + bx + 6$
$P(2) = 0 = 8a + 2b + 6$ ①
$P(-1) = 0 = -a - b + 6$ ②

Solve by adding ① and ②.
$8a + 2b = -6$
$-2a - 2b = -12$ (add)
$6a = -18$
$a = -3$

$$6-2b=-12$$
$$-2b=-18$$
$$b=9$$

$$\therefore P(x)=-3x^3+9x+6$$

b. $P(x)=-3(x^3-3x-2)$

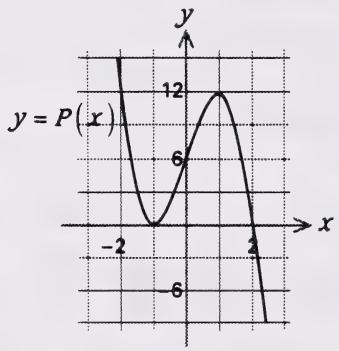
You know that $(x-2)$ is a factor.

-2	1	0	-3	-2
		-2	-4	-2
	1	2	1	0

$$P(x)=-3(x-2)(x^2+2x+1)$$
$$=-3(x-2)(x+1)^2$$

The two given zeros are 2 and -1.
The third zero is also -1.

c.



EXERCISE 6c

9. a. The minimum degree is 4. Both of the roots, at approximately -1.5 and 2, have an even multiplicity. Therefore, there are at least two roots at -1.5 and at least two at 2.

   b. The greatest root is approximately 2. It must be even; thus, it could have multiplicity of two, four, six, eight, ...

   c. The y-intercept is approximately 2; thus, the constant term is also about 2.

   d. To do this you must assume that you know the exact roots of the equation $f(x)=0$. Assuming they are at -1.5 and 2, then there are at least two factors of $(2x+3)$ because of the root of multiplicity two at -1.5, and there are at least two factors of $(x-2)$ because of the root at 2.

   e. $f(x)=k(2x+3)^2(x-2)^2$
   $f(0)=2$
   $\therefore 2=k[2(0)+3]^2(0-2)^2$
   $2=k(9)(4)$
   $2=36k$
   $k=\frac{1}{18}$

   $\therefore f(x)=\frac{1}{18}(2x+3)^2(x-2)^2$

EXERCISE 6d

10. The x-intercepts of the graph are -3, -2 and 1.5 (or $\frac{3}{2}$). So, you know the zeros and the factors of the polynomial function.

$$f(x)=k(x+3)(x+2)(2x-3)$$

Notice that the y-intercept is $(0,36)$. Therefore, $f(0)=36$.

$$36=k(0+3)(0+2)[2(0)-3]$$
$$36=k(3)(2)(-3)$$
$$36=-18k$$
$$k=-2$$
$$f(x)=-2(x+3)(x+2)(2x-3)$$
$$f(x)=-4x^3-14x^2+6x+36$$

11. The x-intercepts are -3 and -2. Because the graph touches the x-axis at -2, this root has a multiplicity of two. Hence, you know the factors of the polynomial function.

$f(x) = k(x+3)(x+2)^2$

You know that $Q(1, 144)$ is on the graph.

So, $f(1) = 144$.

$144 = k(1+3)(1+2)^2$

$144 = k(4)(9)$

$144 = 36k$

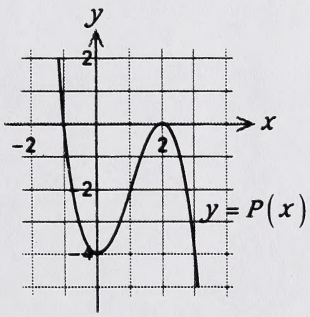
$k = 4$

$f(x) = 4(x+3)(x+2)^2$

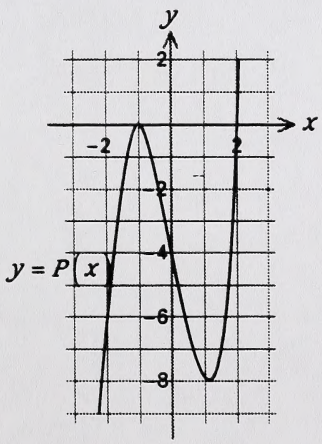
$f(x) = 4x^3 + 28x^2 + 64x + 48$

$\therefore$ The y-intercept of the graph is 48.

12. a. **Graph A**



Graph B



b. In Graph A, 2 is a root of multiplicity 2.

$P(x) = k(x+1)(x-2)^2$

$P(0) = -4$

$\therefore -4 = k(1)(4)$

$k = -1$

$P(x) = -(x+1)(x-2)^2$

$P(x) = -x^3 + 3x^2 - 4$

In Graph B, -1 is a root of multiplicity 2.

$P(x) = k(x+1)^2(x-2)$

$P(0) = -4$

$\therefore -4 = k(1)(-2)$

$k = 2$

$P(x) = 2(x+1)^2(x-2)$

$P(x) = 2x^3 - 6x - 4$

EXERCISE 6e

13. $-x^3 - 2x^2 + 4x + 8 = 8x^3 - 8x^2 - 8x + 8$

$0 = 9x^3 - 6x^2 - 12x$

$0 = 3x^3 - 2x^2 - 4x$

$0 = x(3x^2 - 2x - 4)$

$x = 0$ or $3x^2 - 2x - 4 = 0$

Use the quadratic formula to find the other factors.

$x = 0$ or $x = \dfrac{2 \pm \sqrt{52}}{6}$

$x \doteq 0, -0.87, 1.54$

$P(0) = 8$

$P(-0.87) \doteq 3.67$

$P(1.54) \doteq 5.81$

The points of intersection are $(0, 8)$, $(-0.87, 3.67)$ and $(1.54, 5.81)$.

14. $x^3 - 2x^2 - 4x = -8$

$x^3 - 2x^2 - 4x + 8 = 0$

$x^2(x-2) - 4(x-2) = 0$

$(x^2 - 4)(x-2) = 0$

$(x+2)(x-2)(x-2) = 0$

So, $x = -2$ or 2.

This means that the points $(-2, 8)$ and $(2, 8)$ lie on the graph of $y = F(x)$.

15. $x^3 - 2x^2 - 4x = 0$

$x(x^2 - 2x - 4) = 0$

$x = 0$ and $x = \frac{2 \pm \sqrt{20}}{2}$

$x = 0$ $\quad x \doteq 3.24$ and -1.24

These are the approximate x-intercepts.